The replacement principle in networked economies with single-peaked preferences

Karol Flores-Szwagrzak*

August 20, 2016

Abstract

We study the transfer of a resource from a group of suppliers to a group of demanders through links in a network. The analysis is relevant to situations where institutional constraints bar the use of the price mechanism: the allocation of workloads under fixed salaries, a commodity under disequilibrium prices, etc. In these contexts suppliers and demanders naturally have single-peaked preferences. We evaluate transfer rules on the basis of the "replacement principle" (Thomson, 1997; Moulin, 1987), the requirement that a change in an agent's preferences affects all other agents in the same direction in terms of welfare. We find that the only Pareto-efficient, participation-compatible, replication-invariant, and envy-free rule satisfying an appropriate formulation of the replacement principle is the "egalitarian rule" introduced by Bochet et al. (2012).

Keywords: Replacement principle; Envy-freeness; Single-peaked preferences.
JEL classification: C72; D63; D61; C78; D71

1 Introduction

We study the transfer of a commodity from a group of suppliers to a group of demanders through links in a network. These links model transportation and compatibility

*University of Southern Denmark; karolszw@sam.sdu.dk. First version: December 3rd, 2010. I thank William Thomson for extensive discussions and support. The feedback of an associate editor and two referees significantly improved this paper. I am especially thankful to one of the referees for detailed comments on the proofs and for suggesting explicitly mentioning some of the results now contained in Corollary 1. I gratefully acknowledge the insightful comments of Vikram Manjunath, Hervé Moulin, and Jay Sethuraman. All errors are my own.

constraints or worker-specific qualifications in the distribution of workloads. The analysis is relevant to situations where institutional constraints preclude the use of the price mechanism: the allocation of a commodity under disequilibrium prices, workloads under fixed wages, etc.

The most basic version of the problem studied here was introduced in the seminal work of Sprumont (1991):

> An amount of a divisible commodity is to be distributed among a group of agents who are made better off the closer their assigned amount is to their "peak" or ideal consumption.

More formally, agents are equipped with single-peaked preferences over their possible assignments. For example, if an employee is paid an hourly wage and her disutility of labor is a convex function of labor supplied, her preferences over time worked are single-peaked.

Sprumont's proposed solution to the above resource allocation problem, the "uniform rule," is the only strategy-proof allocation rule recommending efficient and envy-free (Foley, 1967) allocations.[1] In fact, this uniqueness holds even if envy-freeness is weakened to "equal treatment of equals" requiring that identical agents receive identical assignments (Ching, 1994). A considerable literature has confirmed the centrality of the uniform rule, providing various alternative axiomatic rationales for it.[2] We follow Bochet, İlkılıç, Moulin, and Sethuraman (2012) (hereafter BIMS) and Bochet, İlkılıç, and Moulin (2013) (hereafter BIM) in extending this analysis to more general resource allocation problems featuring network constraints.

Concretely, the problem introduced by BIMS and studied here can be posed as follows:

> **Transfer assignment problem** A commodity is to be transferred from a group of sellers to a group of buyers through links in a network. Each seller can only transfer the commodity to buyers directly linked to her. Each buyer has preferences over the total amounts assigned to her (to consume). Each seller has preferences over the total amounts assigned to her (to supply). Preferences are single-peaked.

[1] Strategy-proofness is the requirement on an allocation rule that agents never gain by misreporting their preferences.

[2] See, for instance, Chun (2006), Dagan (1996), Moulin (1999), and Thomson (1994a,b, 1995b, 1997). The prominence of the uniform rule is even confirmed in studies dropping the Pareto-efficiency criterion: see Sönmez (1994) and Kesten (2006). The uniform rule has also been extended to the problem of reallocating a commodity among agents with single-peaked preferences by Thomson (1995a), Klaus et al. (1998), and Kıbrıs and Küçükşenel (2009).

BIMS' proposed solution to the transfer assignment problem, the "egalitarian rule", has remarkable properties: it is the only solution recommending efficient allocations that satisfies "constrained equal treatment of equals" and "voluntary participation" and elicits agents' preferences truthfully. Constrained equal treatment of equals specifies that two buyers (sellers) with the same preferences should receive assignments that are as equal as possible subject to the network constraints. Voluntary participation specifies that each agent should benefit from participating in the transfer of resources, i.e., be ensured an assignment at least as desirable as receiving nothing at all. Thus, the egalitarian rule can be viewed as the extension of Sprumont's uniform rule to the transfer assignment problem.

Our paper complements the work of BIMS by evaluating solutions to the transfer assignment problem on fairness and solidarity grounds. We propose an operationally useful formulation of the "replacement principle" (Thomson, 1997, 2011), the solidarity notion requiring that a change in an agent's preferences affects all other agents in the same direction welfare-wise: they should all be made at least as well off as they were initially or they should all be made at most as well off. As stated, this is too demanding, clashing with basic equity requirements even in Sprumont's basic model (Thomson, 1997). However, qualifying the replacement principle so that it holds provided the changes in preferences are not overly disruptive yields a property that is compatible with various equity requirements.

Our conclusions confirm the importance of BIMS' egalitarian rule. It is the only solution satisfying our qualified version of the replacement principle, efficiency, constrained envy-freeness, voluntary participation, and a very mild "replication invariance" property (Theorem 1). It is also the only solution satisfying efficiency, constrained envy-freeness, voluntary participation and an informational simplicity condition (Theorem 2).

These results can be used to derive further support for the egalitarian rule. An attractive property of the egalitarian rule, "link-monotonicity", specifies how agents are affected by changes in their linkages in the network. As proposed by BIMS, an increase in an agent's links should not be detrimental to the agent's welfare. A consequence of our results is that the egalitarian rule is the only link monotonic rule satisfying our version of the replacement principle, efficiency, constrained envy-freeness, and replication invariance.

The rest of this paper is organized as follows. Section 2 presents the model and background definitions. Section 3 discusses the replacement principle. Section 4 presents our main results, axiomatic characterizations of the egalitarian rule. An Appendix gathers the proofs not included in the body of the paper.

2 Framework

2.1 Model

We formally present the transfer assignment problem in a variable-population framework.[3] There is a set $\mathbb{B}$ of potential buyers and a set $\mathbb{S}$ of potential sellers for a commodity; these sets are countable and disjoint. Groups of sellers and buyers drawn from these sets will be involved in a specific transfer problem; $\boldsymbol{\mathcal{N}}$ denotes the collection of these groups, the finite subsets of $\mathbb{B} \cup \mathbb{S}$ containing at least one buyer and one seller. We refer to buyers and sellers as **agents**.

The requirement that the commodity can only be transferred between specific buyer-seller pairs is modeled by specifying a set of potential transfer partners for each agent: if agent i is a buyer, $\boldsymbol{A_i} \subseteq \mathbb{S}$ denotes the sellers that can potentially provide her with shares of the commodity; if agent i is a seller, $\boldsymbol{A_i} \subseteq \mathbb{B}$ denotes the buyers that can potentially obtain the commodity form her. Let $\boldsymbol{\mathcal{A}}$ denote this class of sets and, for each $N \in \mathcal{N}$, let $\boldsymbol{\mathcal{A}^N}$ denote the set of profiles $A \equiv (A_i)_{i \in N}$ such that, for each $i \in N$, $A_i \in \mathcal{A}$. Each $A \in \mathcal{A}^N$ induces a graph or network consisting of links between sellers and buyers. A link between a seller i and a buyer j is a necessary condition for a transfer of the commodity from i to j and is denoted $\boldsymbol{ij}$ or $\boldsymbol{ji}$; the induced graph is $\boldsymbol{G(A)} \equiv \{ij : i, j \in N, j \in A_i, i \in A_j\}$. Thus, the link ij is formed *bilaterally* when both $j \in A_i$ and $i \in A_j$. For each agent $i \in N$, we denote the agents adjacent to i in $G(A)$ by $\boldsymbol{\Gamma(i; A)}$.[4] For each subgroup $I \subseteq N$, we denote the agents adjacent to an agent in I by $\boldsymbol{\Gamma(I; A)} \equiv \cup_{i \in I} \Gamma(i; A)$.

Agents have preferences over their possible assignments, the amounts transferred from or to them. Each agent i is thus equipped with a preference relation $\boldsymbol{R_i}$ over her possible assignments which we identify with $\mathbb{R}_+$. Moreover, R_i is assumed to be single-peaked: there is a number $\boldsymbol{p(R_i)} \in \mathbb{R}_+$ such that for each pair $x_i, y_i \in \mathbb{R}_+$, if $x_i < y_i \leq p(R_i)$ or $p(R_i) \leq y_i < x_i$, then $y_i \; P_i \; x_i$ where P_i denotes the asymmetric part of R_i. We refer to $p(R_i)$ as the peak of R_i. Let $\boldsymbol{\mathcal{R}}$ denote the class single-peaked preference relations and, for each $N \in \mathcal{N}$, let $\boldsymbol{\mathcal{R}^N}$ denote the class of profiles $R \equiv (R_i)_{i \in N}$ such that, for each $i \in N$, $R_i \in \mathcal{R}$. For each $N \in \mathcal{N}$, an **economy** is the pair $\boldsymbol{e} \equiv (R, A) \in \mathcal{R}^N \times \mathcal{A}^N$; let $\boldsymbol{\mathcal{E}^N} \equiv \mathcal{R}^N \times \mathcal{A}^N$ denote the economies involving

[3] The basic mathematical notation is as follows: If $\{X_i\}_{i \in I}$ is a family of sets indexed by I, X^I denotes the Cartesian product of the these sets taken over I, $\times_{i \in I} X_i$. For each $x \in X^I$ and each $J \subseteq I$, we denote by x_J the projection of x onto X^J. For each pair $x, y \in \mathbb{R}^I$, $x \geq y$ whenever, for each $i \in I$, $x_i \geq y_i$.

[4] More precisely, $\Gamma(i; A) = \{j \in N : ij \in G(A)\}$. Note that in general $\Gamma(i; A) \subseteq A_i$ and that the inclusion may be strict since A_i is a subset of $\mathbb{S} \cup \mathbb{B}$, not necessarily of N. Moreover, links in $G(A)$ are formed bilaterally.

N; for each $R \in \mathcal{R}^N$, let $\boldsymbol{p(R)} \equiv (p(R_i))_{i \in N}$.

For each $N \equiv B \cup S \in \mathcal{N}$, an **allocation** is a list $x \equiv (x_i)_{i \in N} \in \mathbb{R}_+^N$ specifying an assignment for each agent in N. An allocation x is **feasible at $\boldsymbol{A} \in \mathcal{A}^N$** if and only if there is a matrix of nonnegative numbers (x_{ij}) such that, for each $i \in S$ and $j \in B$, (i) $x_{ij} > 0$ only if $ij \in G(A)$, (ii) $x_i = \sum_{j \in B} x_{ij}$, and (iii) $x_j = \sum_{i \in S} x_{ij}$. Let $\boldsymbol{Z(A)}$ denote the set of feasible allocations at A. The following lemma characterizes the feasible allocations without resorting to constructing matrices implementing those allocations. It is usually known as the Supply-Demand Theorem.[5]

Lemma 1 (Feasibility). *Let $N \equiv B \cup S \in \mathcal{N}$ and $A \in \mathcal{A}^N$. Then, the following statements are equivalent:*

(i) x is feasible at A, $x \in Z(A)$.

(ii) For each $B' \subseteq B$, $\sum_{B'} x_j \leq \sum_{\Gamma(B';A)} x_j$ and $\sum_B x_j = \sum_S x_j$.

(iii) For each $S' \subseteq S$, $\sum_{S'} x_j \leq \sum_{\Gamma(S';A)} x_j$ and $\sum_B x_j = \sum_S x_j$.

2.2 Allocation rules and their properties

A **rule** φ is a function that recommends, for each economy (R, A), a feasible allocation, $\boldsymbol{\varphi(R, A)}$ in $Z(A)$. Unless otherwise specified, we state definitions with respect to a group of agents $N \equiv B \cup S \in \mathcal{N}$ and a rule φ.

We now introduce the basic normative properties of rules. We start with the classical efficiency notion. For each $(R, A) \in \mathcal{E}^N$, an allocation $x \in Z(A)$ is (Pareto) **efficient** if there is no $x' \in Z(A)$ such that, for each $i \in N$, $x'_i \; R_i \; x_i$ and, for at least one $i \in N$, $x'_i \; P_i \; x_i$. Let $\boldsymbol{P(R, A)}$ denote the set of *efficient* allocations for (R, A).

Efficiency For each $(R, A) \in \mathcal{E}^N$, $\varphi(R, A) \in P(R, A)$.

Equity and efficiency are the central requirements in the normative theory of resource allocation. The first ordinal operational test of equity for resource allocation is due to Foley (1967): "ask each person to imagine changing places with every other... If no one is willing to change places, the allocation is equitable." The assignments satisfying this requirement have come to be known as "envy-free" and the test as "no-envy."

Here, the use of the no-envy test is not immediate. This test presumes that each agent can compare her assignment to that of any other agent. In some applications

[5] See, for instance, Corollary 2.1.5. in Lovász and Plummer (1986) or Lemma 1 in BIM or BIMS.

of our model this is not the case. For example, a buyer-seller link can model technological compatibility specific to the buyer and seller. The commodity provided by the seller may also have special features that render it useless to other buyers. Thus, a reasonable formulation of "no-envy" for the transfer assignment problem should qualify the no-envy test to account for these sorts of features.

Our first formulation of the no-envy test circumvents the difficulties discussed above. It requires that whenever two agents have the same potential trading partners, they should not envy each other.

Weak no-envy For each $(R, A) \in \mathcal{E}^N$, and each pair $\{i, j\} \subseteq N$ such that $A_i = A_j$, $\varphi_i(R, A)\ R_i\ \varphi_j(R, A)$.

Although this is a natural notion of no-envy it is too weak. There are economies for which its hypothesis are not met and the test is mute. BIMS propose a more demanding fairness test:[6] There should be no agent envying another if her assignment can be improved upon without affecting that of any other agent except the envied agent.

No-envy For each $(R, A) \in \mathcal{E}^N$, each $K \in \{B, S\}$, and each pair $\{i, j\} \subseteq K$, $\varphi_j(R, A)\ P_i\ \varphi_i(R, A)$ implies there is no $x \in Z(A)$ such that for each $k \in N \setminus \{i, j\}$, $\varphi_k(R, A) = x_k$ and $x_i\ P_i\ \varphi_i(R, A)$.

No-envy is a model-free condition. It also coincides with the standard notion *no-envy* in the absence of network constraints, i.e., when every seller is adjacent to every buyer.

The following condition specifies a lower bound on each agent's welfare, that attainable by receiving or transferring non of the commodity. If we interpret this null assignment as an outside option, this condition is akin to individual rationality.

Voluntary participation For each $(R, A) \in \mathcal{E}^N$ and each $i \in N$, $\varphi_i(R, A)\ R_i\ 0$.

The following property specifies how a rule allocates across economies that are replicas of each other. Suppose that we are given two economies such that each agent in one economy has a "clone" in the other economy, an agent with identical preferences and potential transfer partners. Now, suppose that we bring both economies

[6] BIMS formulate this property but it does not appear in their characterization of the egalitarian rule. The equal treatment of equals condition they do use has a similar interpretation. Note that equal treatment of equals is *not* implied by this property (see Proposition 5 in BIMS).

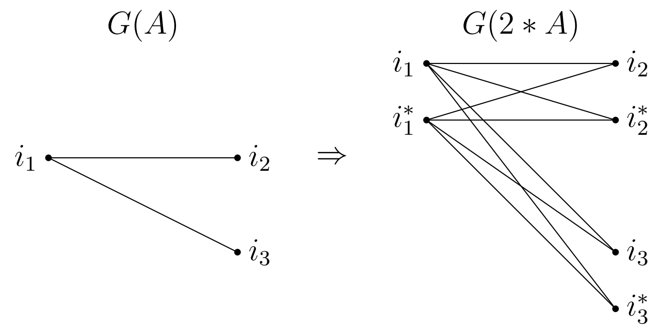

Figure 1: The duplication of an economy $(R, A) \in \mathcal{E}^{B \cup S}$. For each $i_1 \in B$ and each pair $i_2, i_3 \in S$, the restriction of the network $G(2 * A)$ to agents i_1, i_2, i_3, and their clones has the structure shown above.

together. Then, the assignment recommended for each agent in this joint economy is the same as that recommended for her in the original economy that she was a part of. This "replication invariance" property has been studied extensively in the standard division problem with single-peaked preferences and is understood to be very weak.[7]

To formalize the property, let $N \in \mathcal{N}$, $e \equiv (R, A) \in \mathcal{E}^N$, and let $N^* \in \mathcal{N}$ denote a group of agents disjoint from N and with its same cardinality. Then, the economy $\boldsymbol{2 * e} \equiv \boldsymbol{(2 * R, 2 * A)} \in \mathcal{E}^{N \cup N^*}$ is a **duplicate** of e if each agent $i \in N$ has a corresponding "clone" agent $i^* \in N^*$ with identical preferences and potential partners,

$$(2 * R)_i = (2 * R)_{i^*} = R_i \text{ and } (2 * A)_{i^*} = (2 * A)_i = A_i \cup A_{i^*}.$$

For each $x \in \mathbb{R}^N$, let $\boldsymbol{2 * x} \in \mathbb{R}^{N \cup N^*}$ denote the profile such that, for each $i \in N$ and corresponding $i^* \in N^*$, $(2 * x)_i = (2 * x)_{i^*} = x_i$.

Replication invariance For each $e \in \mathcal{E}^N$, $x = \varphi(e)$ implies $2 * x = \varphi(2 * e)$.

The following is an informational simplicity condition.

Peaks-only For each $(R, A) \in \mathcal{E}^N$ and each $R' \in \mathcal{R}^N$ such that $p(R') = p(R)$, $\varphi(R, A) = \varphi(R', A)$.

[7]See Thomson (1994a, 1995, and 1997).

From a practical view-point, *peaks-only* says that agents need only report their preferred transfers. This reduces informational burden of having to collect each agent's full preference relation.

2.3 The egalitarian rule

We now introduce BIMS' egalitarian rule, the first rule proposed and characterized for the transfer assignment problem. This rule physically equalizes assignments as far as is compatible with efficiency and ensuring no agent receives an assignment greater than her peak. This rule equalizes assignments by selecting "Lorenz-dominant" allocations. Let $d \in \mathbb{N}$, $x, y \in \mathbb{R}^d_+$, and let $\hat{x}, \hat{y} \in \mathbb{R}^d_+$ denote the rearrangement of the coordinates of x and y so that $\hat{x}_1 \leq \hat{x}_2 \leq \cdots \leq \hat{x}_d$ and $\hat{y}_1 \leq \hat{y}_2 \leq \cdots \leq \hat{y}_d$, respectively. Then, x **Lorenz-dominates** y, if and only if, for each $k \in \{1, \ldots, d\}$, $\sum_{i=1}^{k} \hat{x}_i \geq \sum_{i=1}^{k} \hat{y}_i$.

Egalitarian rule, *E*: For each $N \equiv B \cup S \in \mathcal{N}$ and each $(R, A) \in \mathcal{E}^N$, $E(R, A)$ is the Lorenz-dominant allocation in $P(R, A) \cap \{z \in \mathbb{R}^N_+ : z \leq p(R)\}$.

This definition highlights the similarity between the egalitarian rule and the uniform rule in the standard division problem with single-peaked preferences of Sprumont (1991). The allocation recommended by the uniform rule is also Lorenz-dominant among all efficient allocations (De Frutos and Massó, 1995). Additionally, the polyhedral structure of $P(R, A) \cap \{z \in \mathbb{R}^N_+ : z \leq p(R)\}$ makes it geometrically isomorphic to the core of a convex transferable utility game. The solution selecting the Lorenz-dominant element in this core is also known as "egalitarian" (Dutta and Ray, 1989).

The following proposition gathers some properties of the egalitarian rule.

Proposition 1. *The egalitarian rule satisfies efficiency, no-envy, voluntary participation, replication invariance, and peaks-only.*

3 The replacement principle

The "replacement principle" states that a change or replacement in the parameters of an allocation problem should affect all the relevant agents in the same direction welfare-wise.[8] The term "replacement" is usually reserved for preference changes.

[8] See Thomson (1999, 2011) for surveys of research on the replacement principle. A replacement principle type property was first introduced by Moulin (1987).

The requirement is that a change in the agent's preferences affects all other agents involved in the allocation problem in the same direction welfare-wise. As formulated by Thomson (1997), the property is:

Welfare-dominance under preference-replacement For each $(R, A) \in \mathcal{E}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, [for each $j \in N \setminus \{i\}, \varphi_j(R, A)\ R_j\ \varphi_j(R'_i, R_{-i}, A)$] or [for each $j \in N \setminus \{i\}, \varphi_j(R'_i, R_{-i}, A)\ R_j\ \varphi_j(R, A)$].

Welfare-dominance under preference-replacement (WDUPR) is a very demanding requirement. It is incompatible with *no-envy* and *efficiency* even in the standard division problem with single-peaked preferences (Thomson, 1997). However, the examples illustrating these incompatibilities are somewhat artificial: the change in an agent's preferences has to be large enough to take an economy where there is scarcity to one where there is too much, or conversely. This may be unrealistic in large scale resource allocation problems. Weakening WDUPR so that it will hold provided preference variations are not this disruptive yields a requirement that is fully compatible with efficiency and equity notions (Thomson, 1997).

The challenge is thus to qualify WDUPR, adapting it to the networked environments studied here. As in Thomson (1997), the key is in formalizing how a change in preferences affects overall scarcity. With network constraints, determining what it means for the commodity to be scarce is not obvious. Since the network $G(A)$ can be an arbitrary bipartite graph, some sellers may not be linked to sufficient buyers to make their desired transfers. Simultaneously, other buyers may not be linked to sufficient supplies.

We are thus interested in arriving at plausible scarcity concepts. The notion we use is that of the *imbalance* between the total demand of a set of buyers and the total supply of the set of sellers who can provide them with the commodity. Formally, consider a set of agents $N \equiv B \cup S \in \mathcal{N}$ and an economy $(R, A) \in \mathcal{E}^N$. Define the **imbalance** of a set $I \subseteq B$ (respectively, of a set $I \subseteq S$)[9] by

$$\textstyle\sum_I p(R_i) - \sum_{\Gamma(I;A)} p(R_i).$$

The imbalance tells us by how much the sum of the peaks of the agents in I falls above or below that of the agents connected to them via the network, $\Gamma(I; A)$. We introduce additional definitions building on this idea. Let $N \equiv B \cup S \in \mathcal{N}$. An economy $(R, A) \in \mathcal{E}^N$ is in

[9]In graph theory, a closely related concept is that of the "deficiency" of a vertex subset in a graph. See Lovasz and Plummer (1986).

buyer-surplus if, for each $I \subseteq S$, $\sum_I p(R_i) \leq \sum_{\Gamma(I;A)} p(R_i)$,

seller-surplus if, for each $I \subseteq B$, $\sum_I p(R_i) \leq \sum_{\Gamma(I;A)} p(R_i)$,

balance if, for each $I \subseteq B$ and each $I \subseteq S$, $\sum_I p(R_i) \leq \sum_{\Gamma(I;A)} p(R_i)$.

The following Lemma enables a *unique* description of the relative scarcity or abundance of the commodity, throughout the corresponding network $G(A)$, in terms of buyer-surplus, seller-surplus, and balance.

Lemma 2 (Decomposition). *Let $N = B \cup S \in \mathcal{N}$ and $(R, A) \in \mathcal{E}^N$. There are unique partitions, $\{B_+, B_0, B_-\}$ of B and $\{S_+, S_0, S_+\}$ of S, respectively, such that*

(i) for each $(K, L) \in \{(B_-, S_-), (B_-, S_0), (B_0, S_-)\}, G(A_{K \cup L}) = \varnothing$,

(ii) $(R_{B_+ \cup S_-}, A_{B_+ \cup S_-})$ has a seller-surplus,

(iii) $(R_{B_- \cup S_+}, A_{B_- \cup S_+})$ has a buyer-surplus,

(iv) $(R_{B_0 \cup S_0}, A_{B_0 \cup S_0})$ is in balance.

See Appendix A.1 for a formal defintion of the partitions in the Lemma. These can be derived from the imbalance notion introduced above. In fact, S_- maximizes the imbalance between supply and demand while B_- maximizes the imbalance between demand and supply. Moreover, the sellers in S_- are only connected to demanders in B_+ and can thus only supply them. The buyers in B_- are only connected to sellers in S_+ and can thus only receive from them. All other buyers and sellers are in B_0 and S_0, respectively. In fact, the Lemma provides a canonical partition of the set of agents into agents who are facing demands exceeding their optimal transfers (those in B_+ and S_+), agents (those in B_- and S_-) who are only connected to agents in B_+ and S_+, and agents who are in neither of the above situations (those in B_0 and S_0).

The Decomposition Lemma is a version of the Gallai-Edmonds Decomposition (GED) of a bipartite graph where it also formalizes a notion of scarcity.[10] Bogomolnaia and Moulin (2004) and Roth et al. (2005) provide applications of the GED to matching under dichotomous preferences and kidney exchange, respectively. In these matching applications, the GED is also used to formalize what agents are difficult to match or facing "scarce" matching prospects.

[10]See, for instance, Schrijver (2003) or Lovász and Plummer (1986) for a mathematical treatment of the Gallai-Edmonds decomposition. Early working paper versions of BIMS relied on the Decomposition Lemma. The closely related work of BIM still uses the Lemma.

For each $N \in \mathcal{N}$ and each $e \equiv (R, A) \in \mathcal{E}^N$, we denote the partition of N into $B_+, B_0, B_-, S_+, S_0, S_-$ in the Decomposition Lemma by $\mathbb{P}(e)$ or $\mathbb{P}(R, A)$.

We are ready to formulate our qualified version of WDUPR for the transfer assignment problem.

Replacement-dominance For each $(R, A) \in \mathcal{E}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, if $\mathbb{P}(R, A) = \mathbb{P}(R'_i, R_{-i}, A)$, [for each $j \in N \setminus \{i\}, \varphi_j(R, A)\ R_j\ \varphi_j(R'_i, R_{-i}, A)$] or [for each $j \in N \setminus \{i\}, \varphi_j(R'_i, R_{-i}, A)\ R_j\ \varphi_j(R, A)$].

This condition coincides with Thomson's (1997) weak WDUPR in the standard division problem with single-peaked preferences introduced by Sprumont (1991). To see this consider an economy (R, A) where each seller is linked to every buyer and where demands exceed supply, $\sum_B p(R_i) \geq \sum_S p(R_i)$. If buyer j's preferences were to change from R_j to $\tilde{R}_j$ and $p(\tilde{R}_j) + \sum_{i \in B \setminus \{j\}} p(R_i) \geq \sum_S p(R_i)$, then *replacement-dominance* requires that each buyer besides j be made at least as well-off as before the change or that each buyer besides j be made at most as well-off. Of course, the same conclusion should follow if we reversed both of the above inequalities.

Proposition 2. *The egalitarian rule is replacement-dominant.*

The notion of scarcity formalized by the Decomposition Lemma is closely related to the structure of the set of *efficient* allocations. In the standard division problem with single-peaked preferences introduced by Sprumont (1991), an allocation is *efficient* if and only if, when the available endowment of the commodity exceeds the aggregate demand for it, every agent receives at least her peak. Similarly, if aggregate demand exceeds the available endowment, no agent receives more than her peak. The following Lemma, characterizing the set of *efficient* allocations, generalizes this relationship to networked environments using our proposed scarcity notion.

Lemma 3 (Efficiency, BIM, BIMS). *Let $N \in \mathcal{N}$, $(R, A) \in \mathcal{E}^N$, and denote the cells of partition $\mathbb{P}(R, A)$ by $B_+, B_0, B_-, S_+, S_0, S_-$. Then, $x \in P(R, A)$ if and only if*

(i) $x_{B_-} \leq p(R_{B_-})$, $x_{S_+} \geq p(R_{S_+})$ *and* $x_{B_- \cup S_+} \in Z(A_{B_- \cup S_+})$,

(ii) $x_{S_-} \leq p(R_{S_-})$, $x_{B_+} \geq p(R_{B_+})$ *and* $x_{B_+ \cup S_-} \in Z(A_{B_+ \cup S_-})$,

(iii) $x_{B_0 \cup S_0} = p(R_{B_0 \cup S_0})$ *and* $x_{B_0 \cup S_0} \in Z(A_{B_0 \cup S_0})$.

4 Normative rationales for the egalitarian rule

Strategy-proofness is central in BIMS characterization of the egalitarian rule. Though it is the most compelling incentive compatibility notion, it is known to be taxing. In general social choice problems, non-trivial strategy-proof social choice functions are dictatorial (Gibbard, 1985; Satterthwaite, 1975). The impossibility of finding more palatable, non-dictatorial, rules extends to more structured economic environments (Hurwicz, 1972; Zhou, 1991; Serizawa, 2002; Goswami et al., 2013). Our main contribution is confirming the centrality of BIMS' egalitarian rule even when we depart from strategy-proofness. We find that replacing strategy-proofness with an ethical requirement - the replacement principle - singles out the egalitarian rule.

Theorem 1. *A rule satisfies efficiency, no-envy, replacement-dominance, replication invariance, and voluntary participation if and only if it is the egalitarian rule.*

Theorem 1 follows from two intermediate results. Theorem 2 (below) establishes that the egalitarian rule is characterized by *efficiency*, *no-envy*, *voluntary-participation*, and *peaks-only*. On the other hand, the following lemma proves that a rule satisfying the properties in Theorem 1 satisfies *peaks-only*.

Lemma 4. *A rule satisfying efficiency, weak no-envy, replacement-dominance, and replication invariance is peaks-only.*

Thus, given Lemma 4, Theorem 2 proves Theorem 1.

Theorem 2. *A rule satisfies efficiency, no-envy, voluntary participation, and peaks-only if and only if it is the egalitarian rule.*

The idea behind the proof of Theorem 2 is to show that, if a rule satisfies all the specified axioms, then its recommended allocation, for each economy, is also the unique solution to the problem of maximizing a separably concave symmetric function over the selection from the set of *efficient* allocations from where the egalitarian rule chooses the Lorenz-dominant element. The argument concludes using a classical result of Hardy, Littlewood, and Polya (1929): the Lorenz-dominant allocation maximizes a separably concave and symmetric function. Hence, any rule satisfying the axioms must in fact coincide with the egalitarian rule.

Though significantly more involved, Theorem 2 can be viewed as an extension of Thomson's (1994b) characterization of the uniform rule. Thomson studied correspondences mapping economies into *sets* of allocations and proved that the uniform rule is the only *peaks-only* correspondence selecting *envy-free* and *efficient* allocations. Our result would also hold for correspondences.

Proof of Theorem 2. By Proposition 1, the egalitarian rule satisfies the properties in Theorem 2. Conversely, let φ denote a rule satisfying properties in Theorem 2. Let $N \equiv B \cup S \in \mathcal{N}$, $(R, A) \in \mathcal{E}^N$, $p \equiv p(R)$, $x \equiv \varphi(R, A)$, $y \equiv E(R, A)$,

$$P^*(R, A) \equiv P(R, A) \cap \{z \in \mathbb{R}^N : z \leq p\}$$

and denote the cells of partition $\mathbb{P}(R, A)$ by $B_+, B_0, B_-, S_+, S_0, S_-$. We will prove that $x = y$.

Step 1. $x \in P^*(R, A)$.

Since φ is *efficient*, $x \in P(R, A)$. Suppose that there is $i \in N$ such that $p_i < x_i$. Let $\tilde{R} \in \mathcal{R}^N$ be such that (i) for each $j \in N \setminus \{i\}$, $\tilde{R}_j = R_j$, and (ii) $\tilde{R}_i$ satisfies $p(\tilde{R}_i) = p(R_i)$ and $0\ \tilde{P}_i\ x_i$. Thus $p(\tilde{R}) = p$. Thus, by *peaks-only*, $\varphi(\tilde{R}, A) = x$ and $0\ \tilde{P}_i\ \varphi_i(\tilde{R}, A)$. This contradicts *volutary participation*.

Step 2. *For each* $z \in P^*(R, A)$ *and each* $i \in B_+ \cup B_0 \cup S_+ \cup S_0$, $z_i = p_i$.

If $z \in P(R, A)$, by the Efficiency Lemma, for each $i \in B_0 \cup S_0$, $z_i = p_i$ and, for each $i \in B_+ \cup S_+$, $z_i \geq p_i$. Additionally, if $z \in P^*(R, A)$, $z \leq p$. Thus, for each $i \in B_+ \cup S_+$, $z_i = p_i$.

The following step establishes that the projection of $P^*(R, A)$ onto $\mathbb{R}^{B_-}$, denoted by $P^*(R, A)|_{B_-}$, can be described in terms of a real-valued sub-modular set function $f : 2^{B_-} \to \mathbb{R}$. This fact underlies our proof that $x_{B_-} = y_{B_-}$.

Step 3. *Define* $f : 2^{B_-} \to \mathbb{R}$ *by,*

$$\text{for each } I \subseteq B_-,\ f(I) \equiv \min\{\textstyle\sum_{\Gamma(J; A_{B_- \cup S_+})} p_i + \sum_{I \setminus J} p_i : J \subseteq I\}.$$

and let $\mathcal{B}(f) \equiv \{z \in \mathbb{R}^{B_-} : \forall I \subseteq B_-, \sum_I z_i \leq f(I); \sum_{B_-} z_i = f(B_-)\}$. *Then,* $\mathcal{B}(f) = P^*(R, A)|_{B_-}$. *Moreover, f is sub-modular, non-decreasing with respect to set inclusion,* $f(\varnothing) = 0$, *and* $\mathcal{B}(f)$ *is compact and convex.*[11]

See the Appendix for a proof of Step 3.

Step 4. *For each* $(i, z) \in B_- \times \mathcal{B}(f)$, *let*

$$\text{dep}(z, i) \equiv \{j \in B_- : \exists \mu \in \mathbb{R}_{++} \ s.t.\ z + \mu(\mathbf{e}_i - \mathbf{e}_j) \in \mathcal{B}(f)\}.^{12}$$

Then, for each $\{i, j\} \subseteq B_-$, *if* $j \in \text{dep}(x_{B_-}, i)$, *then* $x_j \leq x_i$.

[11] A symmetric description of $P^*(R, A)|_{S_-}$ in terms of a sub-modular function is also possible.

[12] As usual $\mathbf{e}_i$ denotes the ith standard basis vector, the vector with a one in the ith coordinate and zeros elsewhere.

By Step 1, $x \in P^*(R,A)$, and, by Step 3, $x_{B_-} \in \mathcal{B}(f)$. Let $\{i,j\} \subseteq B_-$ be such that $j \in \text{dep}(x_{B_-}, i)$. By way of contradiction, suppose that $x_i < x_j$. Let $\tilde{R}_i \in \mathcal{R}$ be such that $p(\tilde{R}_i) = p_i$ and $x_j \; \tilde{P}_i \; x_i$. By *peaks-only*, $\varphi(\tilde{R}_i, R_{-i}, A) = x$. By Step 3 and the definition of $\text{dep}(x_{B_-}, i)$, there is a sufficiently small $\mu > 0$ such that $w \equiv x + \mu(\mathbf{e}_i - \mathbf{e}_j) \in P^*(R,A)$. Since $x_i < w_i \leq p_i$, $w_i \; \tilde{P}_i \; x_i$ and, for each $k \in N \setminus \{i,j\}$, $w_k = x_k$, φ violates *no-envy* at $(\tilde{R}_i, R_{-i}, A)$, a contradiction.

Conclusion By Theorem 9.1 in (Fujishige, 2005), the following conditions are equivalent if f is sub-modular, non-decreasing with respect to set inclusion and $f(\varnothing) = 0$:

- $z \in \mathcal{B}(f)$ is such that, for each $(i,j) \in B_- \times B_-$ such that $j \in \text{dep}(z,i)$, $z_j \leq z_i$.
- $z = \arg\min\{\sum_{B_-} s_i^2 : s \in \mathcal{B}(f)\}$.

Thus, by Step 4, $x_{B_-} = \arg\max\{\sum_{B_-} -s_i^2 : s \in \mathcal{B}(f)\}$. Thus, using the equivalence between (i) and (iv) in Theorem 1 of Schmeidler (1979), x_{B_-}, is Lorenz-dominant over $\mathcal{B}(f)$. However, y_{B_-} is Lorenz-dominant over $\mathcal{B}(f)$ for otherwise, by Step 3, y would not be Lorenz-dominant over $P^*(R,A)$. Since there is a unique Lorenz-dominant element, $y_{B_-} = x_{B_-}$.

A symmetric argument (with analogous versions of Steps 3 and 4) establishes that $y_{S_-} = x_{S_-}$. Thus, by Step 2, $x = y$. ∎

Theorems 1 and 2 can be used to derive further support for the egalitarian rule. We derive further characterizations as corollaries using two properties of the egalitarian rule discussed by BIMS, "link-monotonicity" and "cross-monotonicity."

Link-monotonicity specifies how agents' welfare is affected by changes in their linkages in the network. As proposed by BIMS, an increase (with respect to set inclusion) in an agent's links should not be detrimental to her welfare.

Link-monotonicity For each $(R,A) \in \mathcal{E}^N$, each $i \in N$, and each $A'_i \in \mathcal{A}$ such that $A'_i \subseteq A_i$, $\varphi_i(R,A) \; R_i \; \varphi_i(R, A'_i, A_{-i})$.

Link-monotonicity can be interpreted both normatively and strategically. Normatively, links may encode abilities or specific qualifications that may be attributed to merit. Thus, if an agent's link set increases with respect to set inclusion, this should not hurt her. Strategically, if agents can cut or conceal links involving them, it would be desirable to provide them with incentives not to benefit from doing so.

Cross-monotonicity specifies that, if a seller increases her preferred transfer, no buyer is made worse off by the supply increase while no other seller is made better-off. Similarly, if a buyer increases her preferred transfer, no seller is made worse off by the demand increase while no other buyer is made better off.

Cross-monotonicity For each $N \equiv B \cup S \in \mathcal{N}$, each $(R, A) \in \mathcal{E}^N$, each $K \in \{B, S\}$, each $i \in K$, and each $R'_i \in \mathcal{R}$, if $p(R_i) \leq p(R'_i)$, then [for each $j \in N \setminus K$, $\varphi_j(R'_i, R_{-i}, A)\ R_j\ \varphi_j(R, A)$] and [for each $j \in K \setminus \{i\}$, $\varphi_j(R, A)\ R_j\ \varphi_j(R'_i, R_{-i}, A)$].

Lemma 5. *A rule satisfying efficiency and cross-monotonicity is peaks-only.*

The characterizations in next Corollary now follow from Theorems 1 and 2.

Corollary 1. *The egalitarian rule is the only rule satisfying efficiency, no-envy, and either of the following:*

(i) replacement-dominance, replication invariance, and link-monotonicity;

(ii) peaks-only and link-monotonicity;

(iii) cross-monotonicity and voluntary participation;

(iv) cross-monotonicity and link-monotonicity.

BIMS proved that the egalitarian rule is *link monotonic*. On the other hand, *link-monotonicity* implies *voluntary participation*.[13] Thus, Theorem 1 implies (i), Theorem 2 implies (ii), and Theorem 2 and Lemma 5 imply (iii) and (iv).

[13] To see this, let $N \in \mathcal{N}$, $(R, A) \in \mathcal{E}^N$, and let φ denote a *link monotonic* rule. Then, by *link-monotonicity*, $\varphi_i(R, A)\ R_i\ \varphi_i(R, \varnothing, A_{-i})$. However, by feasibility, $\varphi_i(R, \varnothing, A_{-i}) = 0$ because agent i is not adjacent to any agent in $G(\varnothing, A_{-i})$. Thus, $\varphi_i(R, A)\ R_i\ 0$ and φ satisfies *voluntary praticipation*.

A Appendix

A.1 Lemmas

This Section gathers results used in proving Lemma 4 and that the egalitarian rules are *replication-invariant* and *replacement-dominant*.

For each $N \equiv B \cup S$, each $e \equiv (R, A) \in \mathcal{E}^N$, and each $I \subseteq B$ (each $I \subseteq S$), let

$$\boldsymbol{\delta(I; e)} \equiv \sum_I p(R_j) - \sum_{\Gamma(I;A)} p(R_j).$$

Then, the cells partition $\mathbb{P}(e)$ are defined by:[14]

$$B^e_- \equiv \cap\{I : I \in \arg\max_{J \subseteq B}\{\delta(J; e) : \delta(J; e) > 0\}\}, \quad B^e_+ \equiv \Gamma(S^e_-; A), \quad B^e_0 \equiv B \setminus [B^e_- \cup B^e_+],$$

$$S^e_- \equiv \cap\{I : I \in \arg\max_{J \subseteq S}\{\delta(J; e) : \delta(J; e) > 0\}\}, \quad S^e_+ \equiv \Gamma(B^e_-; A), \quad S^e_0 \equiv S \setminus [S^e_- \cup S^e_+].$$

Remark 1. *For each $N \equiv B \cup S$ and each $e \equiv (R, A) \in \mathcal{E}^N$,*

$$B^e_- \in \arg\max_{J \subseteq B}\{\delta(J; e) : \delta(J; e) > 0\} \quad \text{and} \quad S^e_- \in \arg\max_{J \subseteq S}\{\delta(J; e) : \delta(J; e) > 0\}.$$

That is, B^e_- is contained in every maximizer of $\delta(J; e)$ over $J \subseteq B$ and is itself a maximizer. Similarly, S^e_- is contained in every maximizer of $\delta(J; e)$ over $J \subseteq S$ and is itself a maximizer.

The first result describes the relationship between the decomposition of an economy described in Lemma 2 and that of its duplicate economy.

Lemma 6 (Replication). *For each $N \in \mathcal{N}$, each $e \equiv (R, A) \in \mathcal{E}^N$, and each $k \in \{1, \ldots, |N|\}$,*

$$\begin{aligned} i_k \in B^e_+ &\Leftrightarrow i_k, i^*_k \in B^{2*e}_+ & \quad i_k \in S^e_+ &\Leftrightarrow i_k, i^*_k \in S^{2*e}_+ \\ i_k \in B^e_0 &\Leftrightarrow i_k, i^*_k \in B^{2*e}_0 & \quad i_k \in S^e_0 &\Leftrightarrow i_k, i^*_k \in S^{2*e}_0 \\ i_k \in B^e_- &\Leftrightarrow i_k, i^*_k \in B^{2*e}_- & \quad i_k \in S^e_- &\Leftrightarrow i_k, i^*_k \in S^{2*e}_-. \end{aligned}$$

Proof. Let $N \equiv B \cup S \in \mathcal{E}^N$, $e \equiv (R, A) \in \mathcal{E}^N$, and $\hat{e} \equiv 2 * e$. Let $\hat{N} = \hat{B} \cup \hat{S}$ consist of all the agents involved in economy $\hat{e}$ (the agents in N and their clones). All the cells of $\mathbb{P}(e)$ and $\mathbb{P}(\hat{e})$ are defined in terms of B^e_-, S^e_- and $B^{\hat{e}}_-, S^{\hat{e}}_-$, respectively. Thus, it suffices to show that $i \in B^e_- \Leftrightarrow i, i^* \in B^{\hat{e}}_-$ and that $i \in S^e_- \Leftrightarrow i, i^* \in S^{\hat{e}}_-$.

Let $B' \equiv B^{\hat{e}}_- \cap B$ and $B'' \equiv B^e_- \cup \{i^* : i \in B^e_-\}$. Note that $i \in B^e_- \Leftrightarrow i, i^* \in B^{\hat{e}}_-$ is equivalent to $B' = B^e_-$ which is what we now prove. By Remark 1, $\delta(B^e_-; e) \geq \delta(B'; e)$. Thus, if $\delta(B^e_-; e) > \delta(B'; e)$,

[14]See BIM (proof of Lemma 2) and the closely related result in BIMS (Lemma 2).

$$\delta(B''; \hat{e}) = 2\delta(B^e_-; e) > 2\delta(B'; e) = \delta(B^{\hat{e}}_-; \hat{e}).$$

This contradicts the fact that $B^{\hat{e}}_-$ maximizes $\delta(\cdot; \hat{e})$ over the subsets of $\hat{B}$ (Remark 1). Thus, $\delta(B^e_-; e) = \delta(B'; e)$ and B' is a maximizer of $\delta(\cdot; e)$ over the subsets of B. Since B^e_- is inclusion-minimal across these maximizers, $B^e_- \subseteq B'$. By Remark 1, $\delta(B^{\hat{e}}_-; \hat{e}) \geq \delta(B''; \hat{e})$. Thus, if $\delta(B^{\hat{e}}_-; \hat{e}) > \delta(B''; \hat{e})$,

$$\delta(B'; e) = \tfrac{1}{2}\delta(B^{\hat{e}}_-; \hat{e}) > \tfrac{1}{2}\delta(B''; \hat{e}) = \delta(B^e_-; e).$$

This contradicts the fact that B^e_- maximizes $\delta(\cdot; e)$ over the subsets of B (Remark 1). Thus, $\delta(B^{\hat{e}}_-; \hat{e}) = \delta(B''; \hat{e})$ and B'' is a maximizer of $\delta(\cdot, \hat{e})$ over the subsets of $\hat{B}$. Since $B^{\hat{e}}_-$ is inclusion-minimal across these maximizers, $B^{\hat{e}}_- \subseteq B''$. Thus, $B' = B^{\hat{e}}_- \cap B \subseteq B'' \cap B = B^e_-$. Altogether, $B' = B^e_-$, as desired. A symmetric argument shows that $S^e_- \cup \{i^* : i \in S^e_-\} = S^{\hat{e}}_-$. ∎

Lemma 7. *Let φ be an efficient rule satisfying replacement-dominance. Let $N \in \mathcal{N}$, $e \equiv (R, A) \in \mathcal{E}^N$, and $i \in N$. Let $R'_i \in \mathcal{R}$ be such that $\mathbb{P}(R, A) = \mathbb{P}(R'_i, R_{-i}, A)$. Let $(K, L) \in \{(B^e_-, S^e_+), (S^e_-, B^e_+)\}$, $x \equiv \varphi(R, A)$ and $x' \equiv \varphi(R'_i, R_{-i}, A)$.*

(i) If $i \in K$ and $x'_i \geq x_i$,
$$k \in K \setminus \{i\} \Rightarrow x_k \geq x'_k,$$
$$k \in L \Rightarrow x_k \leq x'_k,$$
$$k \in N \setminus [K \cup L] \Rightarrow x_k = x'_k.$$

(ii) If $i \in L$ and $x'_i \leq x_i$,
$$k \in L \setminus \{i\} \Rightarrow x_k \leq x'_k,$$
$$k \in K \Rightarrow x_k \geq x'_k,$$
$$k \in N \setminus [K \cup L] \Rightarrow x_k = x'_k.$$

Proof. We will prove Statement (i); the proof of Statement (ii) is symmetric. Let all notation be as in the statement of Lemma 7. Let $(K, L) = (B^e_-, S^e_+)$, $i \in K$, and suppose that $x'_i \geq x_i$. By *replacement-dominance*, either

$$[\text{for each } j \in N \setminus \{i\}, x_j \; R_j \; x'_j] \text{ or} \tag{1a}$$
$$[\text{for each } j \in N \setminus \{i\}, x'_j \; R_j \; x_j]. \tag{1b}$$

By the Efficiency Lemma,

$$\begin{aligned} j \in B^e_- \cup S^e_- \quad &\Rightarrow \quad \max\{x_j, x'_j\} \leq p(R_j), \\ j \in B^e_+ \cup S^e_+ \quad &\Rightarrow \quad \min\{x_j, x'_j\} \geq p(R_j), \\ j \in B^e_0 \cup S^e_0 \quad &\Rightarrow \quad x_j = x'_j = p(R_j). \end{aligned} \tag{2}$$

Suppose that (1a) holds. Then, by single-peakedness, (2) implies the desired result:

$$\begin{aligned} j \in B^e_- \cup S^e_- \setminus \{i\} \quad &\Rightarrow \quad x'_j \le x_j \le p(R_j), \\ j \in B^e_+ \cup S^e_+ \quad &\Rightarrow \quad x'_j \ge x_j \ge p(R_j), \\ j \in B^e_0 \cup S^e_0 \quad &\Rightarrow \quad x_j = x'_j = p(R_j). \end{aligned} \tag{3}$$

Suppose instead from here on that (1b) holds. Then, by single-peakedness, (2) implies

$$\begin{aligned} j \in B^e_- \cup S^e_- \setminus \{i\} \quad &\Rightarrow \quad x_j \le x'_j \le p(R_j), \\ j \in B^e_+ \cup S^e_+ \quad &\Rightarrow \quad x_j \ge x'_j \ge p(R_j), \\ j \in B^e_0 \cup S^e_0 \quad &\Rightarrow \quad x_j = x'_j = p(R_j). \end{aligned} \tag{4}$$

By the Efficiency Lemma, $x'_{B^e_- \cup S^e_+}, x_{B^e_- \cup S^e_+} \in Z(A_{B^e_- \cup S^e_+})$. Thus,

$$\sum_{B^e_-} x'_j = \sum_{S^e_+} x'_j \quad \text{and} \quad \sum_{B^e_-} x_j = \sum_{S^e_+} x_j. \tag{5}$$

Case 1: $x'_i = x_i$**.** By (4), $\sum_{B^e_-} x'_j \ge \sum_{B^e_-} x_j$ and $\sum_{S^e_+} x'_j \le \sum_{S^e_+} x_j$. Thus, by (5), $\sum_{B^e_-} x_j = \sum_{B^e_-} x'_j$ and $\sum_{S^e_+} x_j = \sum_{S^e_+} x'_j$. Thus, by (4), $x'_{B^e_- \cup S^e_+} = x_{B^e_- \cup S^e_+}$. Similarly, $x'_{B^e_+ \cup S^e_-} = x_{B^e_+ \cup S^e_-}$.

Case 2: $x'_i > x_i$**.** Then, by (5),

$$[\text{there is } j \in B^e_- \setminus \{i\} \text{ such that } x'_j < x_j \,] \text{ or} \tag{6a}$$

$$[\text{there is } j \in S^e_+ \text{ such that } x'_j > x_j]. \tag{6b}$$

Suppose that (6a) holds. By (2), $x'_j < x_j \le p(R_j)$ which contradicts (4). Suppose that (6b) holds instead. By (2), $x'_j > x_j \ge p(R_j)$ which again contradicts (4). These contradictions imply that we indeed have (3) which is the desired result.

The case in which $(K, L) = (S^e_-, B^e_+)$ is symmetric (exchanging the roles of buyers and sellers). ∎

Proof of Lemma 4. Let φ denote a rule satisfying *weak no-envy*, *efficiency*, *replacement-dominance*, and *replication-invariance*. Let $N \in \mathcal{N}$, $e \equiv (R, A) \in \mathcal{E}^N$, and $x \equiv \varphi(R, A)$. Let $i \in N$ and $R'_i \in \mathcal{R}$ be such that $p(R_i) = p(R'_i)$. Let $x' \equiv \varphi(R'_i, R_{-i}, A)$. We first show that $x' = x$.

Let $\hat{e} \equiv (\hat{R}, \hat{A})$ denote the duplicate economy of e and let $\hat{N}$ denote the agents involved in economy $\hat{e}$ (the agents in N and their clones). Let $\hat{x} \equiv 2 * x$ and

$y \equiv \varphi(R'_i, \hat{R}_{-i}, \hat{A})$. Let $i^* \in \hat{N}$ denote i's clone in $\hat{N} \setminus N$ so that $\hat{R}_i = \hat{R}_{i^*}$ and $\hat{A}_i = \hat{A}_{i^*}$. Suppose that $y_i \neq y_{i^*}$. Then, by *weak no-envy*, either $y_i \neq p(R'_i) = p(\hat{R}_i)$ or $y_{i^*} \neq p(\hat{R}_{i*})$, or both. By the Replication Lemma, i and i^* are in the same cell of partition $\mathbb{P}(\hat{R}, \hat{A})$. By the Decomposition Lemma, since $p(R'_i) = p(\hat{R}_i)$, $\mathbb{P}(\hat{R}, \hat{A}) = \mathbb{P}(R'_i, \hat{R}_{-i}, \hat{A})$ (the decomposition depends on the peaks of a preference profile). Thus, i and i^* are in the same cell of partition $\mathbb{P}(R'_i, \hat{R}_{-i}, \hat{A})$ as well. Thus, by the Efficiency Lemma,

$$\begin{aligned} i \in B^{\hat{e}}_- \cup S^{\hat{e}}_- \quad &\Rightarrow \quad y_i < y_{i^*} \leq p(R'_i) = p(\hat{R}_{i^*}), \text{ or } y_{i^*} < y_i \leq p(R'_i) = p(\hat{R}_{i^*}), \text{ and} \\ i \in B^{\hat{e}}_+ \cup S^{\hat{e}}_+ \quad &\Rightarrow \quad y_i > y_{i^*} \geq p(R'_i) = p(\hat{R}_{i^*}), \text{ or } y_{i^*} > y_i \geq p(R'_i) = p(\hat{R}_{i^*}). \end{aligned}$$

By single-peakedness, either $y_{i^*} \; P'_i \; y_i$ or $y_i \; \hat{P}_{i^*} \; y_{i^*}$. However, $\hat{A}_i = \hat{A}_{i^*}$. This contradicts *weak no-envy*. Thus, $y_i = y_{i^*}$. By Lemma 7, $y = \hat{x}$.

Let $R'_{i^*} \in \mathcal{R}$ be such that $R'_{i^*} = R'_i$. Let $y' \equiv \varphi(R'_i, R'_{i^*}, \hat{R}_{\hat{N}\setminus\{i,i^*\}}, \hat{A})$. Suppose that $y'_i \neq y'_{i^*}$. Then, either $y'_i \neq p(R'_i)$ or $y'_{i^*} \neq p(R'_{i*})$, or both. By the Decomposition Lemma, since $p(R'_{i^*}) = p(\hat{R}_{i^*})$, $\mathbb{P}(\hat{R}, \hat{A}) = \mathbb{P}(R'_i, \hat{R}_{-i}, \hat{A}) = \mathbb{P}(R'_i, R'_{i^*}, \hat{R}_{\hat{N}\setminus\{i,i^*\}}, \hat{A})$. Thus, i and i^* are in the same cell of partition $\mathbb{P}(R'_i, R'_{i^*}, \hat{R}_{\hat{N}\setminus\{i,i^*\}}, \hat{A})$. Thus, by the Efficiency Lemma,

$$\begin{aligned} i \in B^{\hat{e}}_- \cup S^{\hat{e}}_- \quad &\Rightarrow \quad y'_i < y'_{i^*} \leq p(R'_i) = p(R'_{i^*}), \text{ or } y'_{i^*} < y'_i \leq p(R'_i) = p(R'_{i^*}), \text{ and} \\ i \in B^{\hat{e}}_+ \cup S^{\hat{e}}_+ \quad &\Rightarrow \quad y'_i > y'_{i^*} \geq p(R'_i) = p(R'_{i^*}), \text{ or } y'_{i^*} > y'_i \geq p(R'_i) = p(R'_{i^*}). \end{aligned}$$

By single-peakedness, either $y'_{i^*} \; P'_i \; y'_i$ or $y'_i \; P'_{i^*} \; y'_{i^*}$. However, $\hat{A}_i = \hat{A}_{i^*}$. This contradicts *weak no-envy*. Thus, $y'_i = y'_{i^*}$. By Lemma 7, $y' = y = \hat{x}$. Thus, by *replication-invariance*, $x' = x$.

Generally, let $\tilde{R} \in \mathcal{R}^N$ be such that $p(R) = p(\tilde{R})$. Labeling N so that $N = \{1, 2, \ldots, n\}$ and repeating the argument above n-times we obtain,

$$\varphi(R, A) = \varphi(\tilde{R}_1, R_{N\setminus\{1\}}, A) = \varphi(\tilde{R}_1, \tilde{R}_2, R_{N\setminus\{1,2\}}, A) = \cdots = \varphi(\tilde{R}, A). \qquad \blacksquare$$

Proof of Lemma 5. Let φ denote a rule satisfying *efficiency* and *cross-monotonicity*, $N \equiv B \cup S \in \mathcal{N}$, $(R, A) \in \mathcal{E}^N$, $K \in \{B, S\}$, $i \in K$, and $R'_i \in \mathcal{R}$ be such that $p(R_i) = p(R'_i)$. Let $S_-, S_0, S_+, B_-, B_0, B_+$ denote the cells of partition $\mathbb{P}(R, A)$ and $x \equiv \varphi(R, A)$ and $y \equiv \varphi(R'_i, R_{-i}, A)$. Note that $\mathbb{P}(R, A) = \mathbb{P}(R'_i, R_{-i}, A)$ since the profile of peaks is unchanged. Thus, by the Efficiency Lemma,

$$\begin{aligned} &\text{for each } j \in S_0 \cup B_0, \quad x_j = y_j = p(R_j) \\ &\text{for each } j \in S_- \cup B_-, \quad x_j, y_j \leq p(R_j) \\ &\text{for each } j \in S_+ \cup B_+, \quad x_j, y_j \geq p(R_j) \end{aligned} \tag{7}$$

Since $p(R_i) \leq p(R'_i)$, by *cross-monotonicity*, [for each $j \in N \setminus K$, $y_j\ R_j\ x_j$] and [for each $j \in K \setminus \{i\}$, $x_j\ R_j\ y_j$]. Since $p(R_i) \geq p(R'_i)$, by *cross-monotonicity*, [for each $j \in N \setminus K$, $x_j\ R_j\ y_j$] and [for each $j \in K \setminus \{i\}$, $y_j\ R_j\ x_j$]. Thus, each agent $j \in N \setminus \{i\}$ is indifferent between x_j and y_j. Thus, by (7), for each $j \in N \setminus \{i\}$, $x_j = y_j$. Thus, since the sum of seller assignments equals the sum of buyer assignments, $\sum_K x_j = \sum_{N \setminus K} x_j = \sum_{N \setminus K} y_j$, $x_i = y_i$. Thus, $x = y$.

Generally, let $\tilde{R} \in \mathcal{R}^N$ be such that $p(R) = p(\tilde{R})$. Labeling N so that $N = \{1, 2, \ldots, n\}$ and repeating the argument above n-times we obtain,

$$\varphi(R, A) = \varphi(\tilde{R}_1, R_{N \setminus \{1\}}, A) = \varphi(\tilde{R}_1, \tilde{R}_2, R_{N \setminus \{1,2\}}, A) = \cdots = \varphi(\tilde{R}, A). \qquad \blacksquare$$

A.2 Proof of Proposition 1

BIMS establish that the egalitarian rule is *link monotonic*, *efficient*, and that it satisfies *volutnary participation* and *no-envy*.

To show that the egalitarian rule is *peaks-only* observe that, by the Efficiency Lemma, the set of *efficient* allocations for two economies, $(R, A), (R', A') \in \mathcal{E}^N$ such that $p(R) = p(R')$ and $A = A'$, is the same, $P(R, A) = P(R', A')$. Thus, the Lorenz-dominant allocation in $P(R, A) \cap \{z \in \mathbb{R}^N_+ : z \leq p(R)\}$ is the same as that in $P(R', A') \cap \{z \in \mathbb{R}^N_+ : z \leq p(R')\}$. Thus, $E(R, A) = E(R', A)$.

It remains to show that the egalitarian rule is *replication-invariant*.

Claim 1. *The egalitarian rule is replication-invariant.*

Proof. Let $N \in \mathcal{N}$ and $e \equiv (R, A) \in \mathcal{E}^N$. Let $\hat{R} \equiv 2 * R$, $\hat{A} \equiv 2 * A$, and $\hat{e} \equiv (\hat{R}, \hat{A})$. Let $\hat{N}$ denote the agents involved in economy $\hat{e}$ (the agents in N and their clones). Let $x \equiv E(e)$ and $\hat{x} \equiv E(\hat{e})$. It suffices to verify that $\hat{x} = 2 * x$.

By the Efficiency Lemma, all agents in $B^{\hat{e}}_0 \cup S^{\hat{e}}_0 \cup B^{\hat{e}}_+ \cup S^{\hat{e}}_+$, receive at least their peak at $\hat{x}$. By the definition of the egalitarian rule, $\hat{x}$ does not exceed any agent's peak. Thus, for each $i \in B^{\hat{e}}_0 \cup S^{\hat{e}}_0 \cup B^{\hat{e}}_+ \cup S^{\hat{e}}_+$, $\hat{x}_i = p(\hat{R}_i)$. Similarly, for each $i \in B^e_0 \cup S^e_0 \cup B^e_+ \cup S^e_+$, $x_i = p(R_i) = p(\hat{R}_i)$. Thus,

$$\text{for each } i \in B^{\hat{e}}_0 \cup S^{\hat{e}}_0 \cup B^{\hat{e}}_+ \cup S^{\hat{e}}_+,\ \hat{x}_i = p(\hat{R}_i) = (2 * x)_i. \tag{8}$$

It remains to prove that, for each $i \in B^{\hat{e}}_- \cup S^{\hat{e}}_-$, $\hat{x}_i = (2 * x)_i$.

Step 1. *For each $i \in B^{\hat{e}}_- \cup S^{\hat{e}}_-$, $\hat{x}_i = \hat{x}_{i^*}$ where i^* denotes i's clone.*

Otherwise, there is a pair $i, i^* \in B^{\hat{e}}_-$ (or $i, i^* \in S^{\hat{e}}_-$) with $\hat{x}_i \neq \hat{x}_{i^*}$. Without loss of generality, $\hat{x}_i < \hat{x}_{i^*}$. By the Efficiency Lemma, $\hat{x}_i < \hat{x}_{i^*} \leq p(\hat{R}_i) = p(\hat{R}_{i^*})$. Then, because i and i^* are symmetric (same preferences and potential transfer partners) in $\hat{e}$, the allocation obtained by permuting the assignments of i and i^* and leaving all other agents assignments unaltered is in $P(\hat{R}, \hat{A}) \cap \{z \in \mathbb{R}_+^{\hat{N}} : z \leq p(\hat{R})\}$, a convex set.[15] By convexity, this set contains a z such that, for each $j \in \hat{N} \setminus \{i, i^*\}$, $z_j = \hat{x}_j$ and $z_i = \frac{\hat{x}_i + \hat{x}_{i^*}}{2} = z_{i^*}$. Then, z Lorenz-dominates $\hat{x}$. This contradicts the fact that $\hat{x}$ is the Lorenz-dominant element. Thus, for each $i \in B_-^{\hat{e}} \cup S_-^{\hat{e}}$, $\hat{x}_i = \hat{x}_{i^*}$.

Step 2. $\hat{x}_N \in P(R, A) \cap \{z \in \mathbb{R}_+^N : z \leq p(R)\}$.

For each $J \subseteq B_-^e$ let $J^* \equiv \{j^* \in \hat{N} : j^* \text{ is the clone of } j \in J\}$. By the Replication Lemma $J^* \subseteq B_-^{\hat{e}}$. By the Efficiency Lemma, $\hat{x}_{B_-^{\hat{e}} \cup S_+^{\hat{e}}} \in Z(\hat{A}_{B_-^{\hat{e}} \cup S_+^{\hat{e}}})$. Thus, by the Feasibility Lemma and Step 1,

$$2 \cdot \sum_J \hat{x}_i = \sum_J \hat{x}_i + \sum_{J^*} \hat{x}_{i^*} \leq \sum_{\Gamma(J \cup J^*; \hat{A}_{B_-^{\hat{e}} \cup S_+^{\hat{e}}})} \hat{x}_i = 2 \cdot \sum_{\Gamma(J; A_{B_-^e \cup S_+^e})} \hat{x}_i$$

$$\text{and} \quad 2 \cdot \sum_{B_-^e} \hat{x}_i = \sum_{B_-^{\hat{e}}} \hat{x}_i = \sum_{S_+^{\hat{e}}} \hat{x}_i = 2 \cdot \sum_{S_+^e} \hat{x}_i.$$

Thus, for each $J \subseteq B_-^e$, $\sum_J \hat{x}_i \leq \sum_{\Gamma(J; A_{B_-^e \cup S_+^e})} p(R_i)$ and $\sum_{B_-^e} \hat{x}_i = \sum_{S_+^e} \hat{x}_i$. By the Feasibility Lemma, $\hat{x}_{B_-^e \cup S_+^e} \in Z(A_{B_-^e \cup S_+^e})$. Similarly, $\hat{x}_{B_+^e \cup S_-^e} \in Z(A_{B_+^e \cup S_-^e})$, and by (8) and the Efficiency Lemma, $\hat{x}_{B_0^e \cup S_0^e} \in Z(A_{B_0^e \cup S_0^e})$. Thus, by (8) and the fact that $\hat{x}_N \leq p(\hat{R})|_N = p(R)$ we have $\hat{x}_N \in P(R, A) \cap \{z \in \mathbb{R}_+^N : z \leq p(R)\}$.

Step 3. $\hat{x} = 2 * x$.

Recall that $N^* \equiv \hat{N} \setminus N$ and that, for each $i \in N$ and her clone $i^* \in N^*$, $A_i = \hat{A}_i = \hat{A}_{i^*}$. Thus, $(2 * x)_N = x \in Z(A)$ implies $(2 * x)_{N^*} \in Z(\hat{A}_{N^*})$. Thus, because $Z(\hat{A}_N) \times Z(\hat{A}_{N^*}) \subseteq Z(\hat{A})$, $2 * x \in Z(\hat{A})$. Moreover, by Lemma 6, (8), and $2 * x \leq p(\hat{R})$, $2 * x \in P(\hat{R}, \hat{A}) \cap \{z \in \mathbb{R}_+^{\hat{N}} : z \leq p(\hat{R})\}$. Thus, since $\hat{x}$ is the egalitarian allocation at $(\hat{R}, \hat{A})$, it Lorenz-dominates $2 * x$. However, by the definition of $2 * x$ and by Step 1, for each $i \in N$ and corresponding clone i^*, $(2 * x)_i = (2 * x)_{i^*}$ and $\hat{x}_i = \hat{x}_{i^*}$. Thus, $\hat{x}_N$ Lorenz-dominates x. On the other hand, since x is the egalitarian allocation for (R, A), by Step 2, x Lorenz-dominates $\hat{x}_N$. Since there is only one Lorenz-dominant allocation in $P(R, A) \cap \{z \in \mathbb{R}_+^N : z \leq p(R)\}$, x and $\hat{x}_N$ coincide. Thus, by Step 1 again, $2 * x$ and $\hat{x}$ coincide. ∎

[15] Convexity follows from the Efficiency and Feasibiltiy Lemmas. From these we deduce that $P(\hat{R}, \hat{A})$ can be described as the intersection of closed half-spaces in $\mathbb{R}_+^{\hat{N}}$ and is hence convex. Since $\{z \in \mathbb{R}_+^{\hat{N}} : z \leq p(\hat{R})\}$ is convex so is the intersection.

A.3 Proof of Proposition 2

We will use the fact that the egalitarian rule is *cross-monotonic* (Bochet et al., 2012) to prove that it is *replacement-dominant*. Let $N \equiv B \cup S \in \mathcal{N}$, $A \in \mathcal{A}^N$, $R \in \mathcal{R}^N$ and $i \in N$. Let $R' \in \mathcal{R}^N$ be such that, for each $j \in N \setminus \{i\}$, $R'_j = R_j$ and suppose, in accordance with the hypothesis of *replacement-dominance*, the partitions $\mathbb{P}(R, A)$ and $\mathbb{P}(R', A)$ coincide. Denote the common cells of these partitions by $S_-, S_+, S_0, B_-, B_+, B_0$. and let $x \equiv E(R, A)$ and $y \equiv E(R', A)$. The egalitarian rule recommends *efficient* allocations under which no agent's assignment exceeds her peak. Thus, by the Efficiency Lemma, for each $j \in S_+ \cup S_0 \cup B_+ \cup B_0$, $x_j = p(R_j)$ and $y_j = p(R'_j)$. Since the egalitarian rule is *peaks-only*, if $p(R_i) = p(R'_i)$, then $x = y$ and *replacement-dominance* is satisfied. If instead $p(R_i) \neq p(R'_i)$, $i \notin B_0 \cup S_0$: by the Efficiency Lemma, $\sum_{B_0} p(R_i) = \sum_{S_0} p(R_i)$ and $\sum_{B_0} p(R'_i) = \sum_{S_0} p(R'_i)$ and these conditions could not hold if $p(R_i) \neq p(R'_i)$. Thus,

$$\begin{aligned} &\text{for each } j \in S_+ \cup B_+,\ x_j = p(R_j) \text{ and } y_j = p(R'_j);\\ &\text{for each } j \in B_0 \cup S_0,\ x_j = p(R_j) = p(R'_j) = y_j. \end{aligned} \tag{9}$$

Suppose that $p(R_i) < p(R'_i)$ and $i \in S$. By *cross-monotonicity*, for each $j \in B$, $y_j\ R_j\ x_j$ and, for each $j \in S \setminus \{i\}$, $x_j\ R_j\ y_j$. Thus, since $x \leq p(R)$ and $y \leq p(R')$, $x_B \leq y_B$ and, $x_{S\setminus\{i\}} \geq y_{S\setminus\{i\}}$.

- If $i \in S_-$, by (9), $y_{S_+} = p(R_{S_+}) = x_{S_+}$. By the Efficiency Lemma, the coordinates of y_{B_-} and x_{B_-} thus add up to $\sum_{S_+} p(R_j)$. Thus, since $x_{B_-} \leq y_{B_-}$, $y_{B_-} = x_{B_-}$. Thus x and y coincide on $N \setminus S_-$. Thus, for each $j \in N \setminus \{i\}$, $x_j\ R_j\ y_j$.

- If $i \in S_+$, by (9), $y_{B_+} = p(R_{B_+}) = x_{B_+}$. By the Efficiency Lemma, the coordinates of y_{S_-} and x_{S_-} thus add up to $\sum_{B_+} p(R_j)$. Thus, since $x_{S_-} \geq y_{S_-}$, $y_{S_-} = x_{S_-}$. Thus x and y coincide on $N \setminus B_-$. Thus, for each $j \in N \setminus \{i\}$, $y_j\ R_j\ x_j$.

If instead $p(R_i) > p(R'_i)$ and $i \in S$, similar arguments establish that each agent other than i is affected in the same direction welfare-wise. The arguments establishing this when $i \in B$ are symmetric. ∎

A.4 Proof of Step 3 in Theorem 2

We first prove that $\mathcal{B}(f) = P^*(R, A)|_{B_-}$. Let $w \in \mathcal{B}(f)$. Then, for each $I \subseteq B_-$, $\sum_I w_i \leq f(I) \leq \sum_{\Gamma(I;A_{B_- \cup S_+})} p_i$ and, for each $i \in B_-$, $w_i \leq f(\{i\}) \leq p_i$.

Additionally, by the definition of B_- in the beginning of Section A.1, $f(B_-) = \sum_{\Gamma(B_-;A_{B_- \cup S_+})} p_i = \sum_{S_+} p_i$. Thus, $\sum_{B_-} w_i = \sum_{S_+} p_i$. Thus, by the Feasibility and Efficiency Lemmas, $w \in P^*(R,A)|_{B_-}$. Conversely, let $w \in P^*(R,A)|_{B_-}$. By the Feasibility Lemma,

$$\text{for each } I \subseteq B_-, \sum_I w_i \leq \sum_{\Gamma(I;A_{B_- \cup S_+})} p_i \text{ and } \sum_{B_-} w_i = \sum_{S_+} p_i.$$

Additionally, because $w \leq p_{B_-}$, for each $I \subseteq B_-$, $\sum_I w_i \leq \sum_I p_i$. Thus, for each pair J, K such that $J \subseteq K \subseteq B_-$, $\sum_J w_i \leq \sum_{\Gamma(J;A_{B_- \cup S_+})} p_i$ and $\sum_{K \setminus J} w_i \leq \sum_{K \setminus J} p_i$. Thus, $\sum_K w_i = \sum_J w_i + \sum_{K \setminus J} w_i \leq \sum_{\Gamma(J;A_{B_- \cup S_+})} p_i + \sum_{K \setminus J} p_i$. Thus, for each $K \subseteq B_-$, $\sum_K w_i \leq f(K)$ and, as remarked above, $\sum_{S_+} p_i = f(B_-)$. Thus, $w \in \mathcal{B}(f)$.

We now prove that f is sub-modular. Let $I_1, I_2 \subseteq B_-$. Let $J_1 \subseteq I_1$ and $J_2 \subseteq I_2$ attain the minima in the definition of f, for I_1 and I_2, respectively. Then, by the sub-modularity of the mapping $I \mapsto \sum_{\Gamma(I;A_{B_- \cup S_+})} p_i$ and the definition of f,

$$\begin{aligned} f(I_1) + f(I_2) &= \sum_{\Gamma(J_1;A_{B_- \cup S_+})} p_i + \sum_{I_1 \setminus J_1} p_i + \sum_{\Gamma(J_2;A_{B_- \cup S_+})} p_i + \sum_{I_2 \setminus J_2} p_i \\ &\geq \sum_{\Gamma(J_1 \cup J_2;A_{B_- \cup S_+})} p_i + \sum_{I_1 \setminus J_1} p_i + \sum_{\Gamma(J_1 \cap J_2;A_{B_- \cup S_+})} p_i + \sum_{I_2 \setminus J_2} p_i \\ &= \sum_{\Gamma(J_1 \cup J_2;A_{B_- \cup S_+})} p_i + \sum_{[I_1 \cup I_2] \setminus [J_1 \cup J_2]} p_i + \sum_{\Gamma(J_1 \cap J_2;A_{B_- \cup S_+})} p_i + \sum_{[I_1 \cap I_2] \setminus [J_1 \cap J_2]} p_i \\ &\geq f(I_1 \cup I_2) + f(I_1 \cap I_2). \end{aligned}$$

Since $I_1, I_2 \subseteq B_-$ were chosen arbitrarily, f is sub-modular.

We now prove that f is non-decreasing. Let $I_1 \subseteq I_2 \subseteq B_-$. Let $J_1 \subseteq I_1$ and $J_2 \subseteq I_2$ attain the minima in the definition of f, for I_1 and I_2, respectively. Then,

$$\begin{aligned} f(I_1) = \sum_{\Gamma(J_1;A_{B_- \cup S_+})} p_i + \sum_{I_1 \setminus J_1} p_i &\leq \sum_{\Gamma([J_2 \cap I_1];A_{B_- \cup S_+})} p_i + \sum_{I_1 \setminus [J_2 \cap I_1]} p_i \\ &\leq \sum_{\Gamma(J_2;A_{B_- \cup S_+})} p_i + \sum_{I_2 \setminus J_2} p_i = f(I_2) \end{aligned}$$

where the first inequality follows from the definition of J_1, and the second one from the fact that $\Gamma([J_2 \cap I_1]; A_{B_- \cup S_+}) \subseteq \Gamma(J_2; A_{B_- \cup S_+})$ and $I_1 \setminus [J_2 \cap I_1] \subseteq I_2 \setminus J_2$. Thus, $f(I_1) \leq f(I_2)$ and f is non-decreasing. Note that $f(\varnothing) = 0$ is immediate from the definition of f.

We now prove that $\mathcal{B}(f)$ is compact and convex. First note that, for each $z \in \mathcal{B}(f)$ and each $i \in B_-$, $z_i \geq 0$.[16] Moreover, since p is a profile of non-negative finite real numbers, for each $I \subseteq B_-$, $f(I) < \infty$. Thus, $\mathcal{B}(f)$ is bounded. Since $\mathcal{B}(f)$ is defined as the intersection of finitely many closed half spaces (weak inequalities) it is convex and closed. Since $\mathcal{B}(f)$ is closed and bounded, it is compact.

A.5 Independence of the axioms

For simplicity, we establish the independence of the axioms in our theorems for economies where each seller is linked to each buyer. We refer to such economies as **connected**. The main results in Thomson (1994b) and Thomson (1995b) on the standard division problem with single-peaked preferences are obtained under the assumption that, for each agent, there is a finite assignment indifferent to 0. If $R_i \in \mathcal{R}$ satisfies this restriction, $\boldsymbol{d(R_i)} \equiv \max\{d \in \mathbb{R}_+ : d \; R_i \; 0\}$ is well defined and finite. For simplicity, we will work under this restriction throughout this section. Note that all of our results hold under this restriction since we never relied on preferences not satisfying it in our analysis.

We will use the uniform rule for the standard division problem with single-peaked preferences to construct a family of rules illustrating the independence of the axioms. For each finite set of buyers or sellers K and each $R_K \equiv (R_i)_{i \in K} \in \mathcal{R}^K$, define:

$\boldsymbol{U(R_K, m)} \in \mathbb{R}^K_+$ where $m \in \mathbb{R}_+$ and, for each $i \in K$,

$$U_i(R_K, m) = \begin{cases} \min\{\lambda, p(R_i)\} & \text{if} \quad \sum_{i \in K} p(R_i) \geq m \\ \max\{\lambda, p(R_i)\} & \text{if} \quad \sum_{i \in K} p(R_i) \leq m \end{cases}$$

where $\lambda \in \mathbb{R}_+$ is such that $\sum_{i \in K} U_i(R, m) = m$.

$\boldsymbol{\sigma(R_K)} \equiv \sum_{i \in K} p(R_i)$.

Let f denote a function mapping preference profiles into a non-negative real numbers. The rule associated with f, denoted by $\boldsymbol{E^f}$, is such that, for each $N \equiv B \cup S \in \mathcal{N}$, each connected $(R, A) \in \mathcal{E}^N$, and each $i \in N$,

$$E^f_i(R, A) = \begin{cases} U_i(R_S, m) & \text{if} \quad i \in S \\ U_i(R_B, m) & \text{if} \quad i \in B \end{cases} \qquad \text{where} \quad m = \text{median}\{\sigma(R_B), \sigma(R_S), f(R)\}.$$

Lemma 8. *Rule E^f satisfies efficiency and no-envy in connected economies.*

[16] Note that $\sum_{B_-} z_j = f(B_-)$ and, for each $i \in B_-$, $\sum_{B_- \setminus i} z_j \leq f(B_- \setminus i)$. Thus, because f is non-decreasing, $z_i \geq f(B_-) - f(B_- \setminus i) \geq 0$.

Proof. Let f denote a function mapping preference profiles into a non-negative real numbers. Let $N \equiv B \cup S \in \mathcal{N}$, $(R, A) \in \mathcal{E}^N$ be connected, and $x \equiv E^f(R, A)$. By definition, if $m \equiv \text{median}\{\sigma(R_B), \sigma(R_S), f(R)\}$,

$$\textstyle\sum_{i \in B} x_i = \sum_{i \in B} U_i(R_B, m) = m = \sum_{i \in S} U_i(R_S, m) = \sum_{i \in S} x_i.$$

Thus, x is feasible at A. Without loss of generality, suppose that $\sigma(R_B) \geq \sigma(R_S)$. Since all sellers are linked to all buyers and conversely, (R, A) is in buyer-surplus; additionally, $\sigma(R_B) \geq m \geq \sigma(R_S)$ which implies that,

$$x_i = \begin{cases} U_i(R_S, m) \geq p(R_i) & \text{if} \quad i \in S \\ U_i(R_B, m) \leq p(R_i) & \text{if} \quad i \in B \end{cases}$$

Thus, by Lemmas 2 and 3, $x \in P(R, A)$, establishing that E^f recommends feasible and efficient allocations. Note that x_B is the allocation recommended by the uniform rule for the standard division problem with single-preferences when the preference profile is R_B and the amount to allocate is m; since the uniform rule satisfies no-envy in that setting (Sprumont, 1991), for each $\{i, j\} \subseteq B$, $x_i \; R_i \; x_j$. Similarly, for each $\{i, j\} \subseteq S$, $x_i \; R_i \; x_j$. Thus, E^f satisfies *no-envy*. ∎

For each $N \equiv B \cup S \in \mathcal{N}$ and each connected $(R, A) \in \mathcal{E}^N$, define the following rules:

$\boldsymbol{\varphi^1}$ is such that, for each $i \in N$, $\varphi^1_i(R, A) = 0$.

$\boldsymbol{\varphi^2}$ is such that:

(i) if $\min\{\sigma(R_B), \sigma(R_S)\} = 0$, then, for each $i \in N$, $\varphi^2_i(R, A) = 0$;

(ii) if $\min\{\sigma(R_B), \sigma(R_S)\} > 0$, then,

$$\varphi^2_i(R, A) = \begin{cases} p(R_i) \frac{\min\{\sigma(R_B), \sigma(R_S)\}}{\sigma(R_S)} & \text{if} \quad i \in S \\ \\ p(R_i) \frac{\min\{\sigma(R_B), \sigma(R_S)\}}{\sigma(R_B)} & \text{if} \quad i \in B \end{cases}.$$

$\boldsymbol{\varphi^3} = E^f$ where

$$f(R) = \begin{cases} |S| \cdot \min_{i \in S} d(R_i) & \text{if } \sigma(R_S) < \sigma(R_B) \\ |B| \cdot \min_{i \in B} d(R_i) & \text{otherwise} \end{cases}.$$

$\boldsymbol{\varphi^4} = E^f$ where

$$f(R) = \begin{cases} d(R_i) & \text{if } \exists i \in S \text{ s.t. } S = \{i\} \text{ and } \sigma(R_S) < \sigma(R_B) \\ 0 & \text{otherwise} \end{cases}.$$

$\boldsymbol{\varphi^5} = E^f$ where $f(R) = \sigma(R_S)$.

Table 1 summarizes the axioms satisfied and not satisfied by these rules.

	φ^1	φ^2	φ^3	φ^4	φ^5
efficiency	no	yes	yes	yes	yes
no-envy	yes	no	yes	yes	yes
replacement-dominance	yes	yes	no	yes	yes
replication invariance	yes	yes	yes	no	yes
voluntary participation	yes	yes	yes	yes	no
peaks-only	yes	yes	no	no	yes

Table 1: A rule satisfies (yes) or does not (no) one of the axioms (Proposition 3).

Proposition 3. *(i) Rules $\varphi^2, \varphi^3, \varphi^4$, and φ^5 satisfy* efficiency, *φ^1 does not. (ii) Rules $\varphi^1, \varphi^3, \varphi^4$, and φ^5 satisfy* no-envy, *φ^2 does not. (iii) Rules $\varphi^1, \varphi^2, \varphi^4$, and φ^5 satisfy* replacement-dominance, *φ^3 does not. (iv) Rules $\varphi^1, \varphi^2, \varphi^3$, and φ^5 satisfy* replication invariance, *φ^4 does not. (v) Rules φ^1, φ^2, φ^3, and φ^4 satisfy* voluntary trade, *φ^5 does not. (vi) Rules φ^1, φ^2, and φ^5 satisfy* peaks-only, *φ^3 and φ^4 do not.*

Proof. (i) By Lemma 8, rules φ^3, φ^4, and φ^5 satisfy *efficiency*. To see that φ^2 does so as well, let $N \equiv B \cup S \in \mathcal{N}$, $(R, A) \in \mathcal{E}^N$, and $x \equiv \varphi^2(R, A)$. Without loss of generality, assume that $\sigma(R_S) \leq \sigma(R_B)$. Thus, $\sum_B x_i = \sigma(R_S) = \sum_S x_i$. Thus, since, for each $i \in S$, $x_i \leq p(R_i)$, $x_i = p(R_i)$. Since, for each $i \in B$, $x_i \leq p(R_i)$, an allocation Pareto-improving upon x would require reallocating the amount $\sigma(R_S)$ among the buyers; however, reallocating would entail reducing the assignment of a buyer, making her worse off.

(ii) By Lemma 8, rules φ^3, φ^4, and φ^5 satisfy *no-envy*. Since, for each economy and each agent, φ^1 recommends the same assignment, this rule satisfies *no-envy* as well. To see that φ^2 does not, let $B = \{i, j\}$, $S = \{k, l\}$, and $e_1 \equiv (R, A) \in \mathcal{E}^{B \cup S}$ be such that $p(R_i) = 4$, $p(R_j) = 2$, $p(R_k) = p(R_l) = 1$, and $x \equiv \varphi^2(R, A)$. Then, $x_i = \frac{4}{3}$, $x_j = \frac{2}{3}$, and $x_k = 1 = x_l$. Thus, $x_i \; P_j \; x_j$ while the allocation y such that $y_i = 1 = y_j$, and $y_k = 1 = y_l$ is also feasible, in violation of *no-envy*.

(iii) Since φ^1 recommends the same allocation for each economy, it satisfies *replacement-dominance*. We now establish that rules φ^2 and φ^5 do so as well. Let $N \equiv B \cup S \in \mathcal{N}$, $(R, A) \in \mathcal{E}^N$, $i \in N$, and $R' \in \mathcal{R}^N$ be such that, for each $j \in N \setminus \{i\}$, $R_j = R'_j$. Suppose that $\mathbb{P}(R, A) = \mathbb{P}(R', A)$ so that *replacement-dominance* applies.

If $\sigma(R_B) \geq \sigma(R_S)$, then $\mathbb{P}(R, A) = \mathbb{P}(R', A)$ implies that $\sigma(R'_B) \geq \sigma(R'_S)$. We now establish that each agent other than i is affected in the same direction welfare-wise under rules φ^2 and φ^5 when the preferences of i change from R_i to R'_i:

- If $j \in S \setminus \{i\}$, $\varphi_j^2(R,A) = p(R_j) = p(R'_j) = \varphi_j^2(R',A)$. Thus, each $j \in S \setminus \{i\}$ is indifferent between the assignment she receives in (R,A) and (R',A). For each $j \in B \setminus \{i\}$, $\varphi_j^2(R,A) = \frac{\sigma(R_S)}{\sigma(R_B)} p(R_j)$ and $\varphi_j^2(R',A) = \frac{\sigma(R'_S)}{\sigma(R'_B)} p(R_j)$. Thus, if $\frac{\sigma(R_S)}{\sigma(R_B)} \leq (\geq) \frac{\sigma(R'_S)}{\sigma(R'_B)}$, each $j \in B \setminus \{i\}$ is at least (at most) as well-off. Thus, all agents other than i are at least (at most) as well-off.

- If $j \in S \setminus \{i\}$, $\varphi_j^5(R,A) = p(R_j) = p(R'_j) = \varphi_j^5(R',A)$. Thus, each $j \in S \setminus \{i\}$ is indifferent between the assignment she receives in (R,A) and (R',A).

  If $i \in S$ the amount transferred from sellers to buyers changes from $\sigma(R_S)$ to $\sigma(R'_S)$. By the properties of the uniform rule (one-sided resource-monotonicity in Thomson, 1994b), $\sigma(R_S) \leq (\geq) \sigma(R'_S)$ implies each buyer's assignment increases (decreases) or remains constant while being bounded above by her peak as was the case before the preference change, thus making each buyer at least (at most) as well-off.

  If $i \in B$ the amount transferred from sellers to buyers remains constant. By the properties of the uniform rule (one-sided welfare domination under preference replacement in Thomson, 1997), $p(R_i) \geq (\leq) p(R'_i)$ implies that, for each buyer other that i, her assignment (decreases) or remains constant while being bounded above by her peak as was the case before the preference change. This makes each buyer other than i at least (at most) as well-off.

  Thus, all agents other than i are at least (at most) as well-off.

If $\sigma(R_B) < \sigma(R_S)$, similar arguments establish that each agent other than i is affected in the same direction welfare-wise under rules φ^2 and φ^5 in response to the preference change. These rules thus satisfy *replacement-dominance*.

We now prove that φ^4 satisfies *replacement-dominance*. Note that, if $|S| \geq 2$, φ^4 coincides with the egalitarian rule since $E = E^f$ when $f = 0$. Thus, φ^4 satisfies *replacement-dominance* in economies where $|S| \geq 2$.

Thus, it remains to consider the case where S is a singleton: let $N \equiv B \cup S \in \mathcal{N}$ where S is the singleton $\{k\}$, $(R,A) \in \mathcal{E}^N$, $i \in N$, and $R' \in \mathcal{R}^N$ be such that, for each $j \in N \setminus \{i\}$, $R_j = R'_j$. Suppose that $\mathbb{P}(R,A) = \mathbb{P}(R',A)$ so that *replacement-dominance* applies. We now determine how the welfare of agents other than i is affected in going from $\varphi^4(R,A)$ to $\varphi^4(R',A)$.

- Case 1: $\sigma(R_S) \geq \sigma(R_B)$. Then $\mathbb{P}(R,A) = \mathbb{P}(R',A)$ implies $\sigma(R'_S) \geq \sigma(R'_B)$ and $f(R) = 0 = f(R')$. Thus, under preference profile R, the amount transferred from the seller to the buyers is $\sigma(R_B)$, while, under preference profile R', the amount is $\sigma(R'_B)$.

Case 1.1: $i = k$. Then, $R_B = R'_B$ and $\sigma(R_B) = \sigma(R'_B)$. Thus, under both preference profiles, the amount transferred to buyers is the sum of their peaks and each buyer's assignment is equal to her peak. This makes each buyer equally well-off under both preference profiles.

Case 1.2: $i \in B$. The amount transferred from the seller to the buyers goes from $\sigma(R_B)$ to $\sigma(R'_B)$. By the properties of the uniform rule (one-sided welfare domination under preference replacement in Thomson, 1997), for each buyer other that i, her assignment increases (decreases) or remains constant while being bounded below by her peak, thus making each buyer other than i at least (at most) as well-off.

- Case 2: $\sigma(R_S) < \sigma(R_B)$. Then, $\mathbb{P}(R, A) = \mathbb{P}(R', A)$ implies $\sigma(R'_S) < \sigma(R'_B)$. The amount transferred from the seller to the buyers goes from $\min\{\sigma(R_B), d(R_k)\}$ to $\min\{\sigma(R'_B), d(R'_k)\}$.

Case 2.1: $i = k$. Then, $\sigma(R_B) = \sigma(R'_B)$. Thus, if $d(R_k) \leq(\geq)\ d(R'_k)$, the amount transferred by the seller is at least (at most) as large. By the properties of the uniform rule (one-sided resource-monotonicity in Thomson, 1994b), each buyer's assignment increases (decreases) or remains constant while being bounded above by her peak, thus making each buyer at least (at most) as well-off.

Case 2.2: $i \in B$. Then, $d(R_k) = d(R'_k)$.

Case 2.2.1: $\min\{\sigma(R_B), d(R_k)\} = d(R_k) = \min\{\sigma(R'_B), d(R_k)\}$ so the amount transferred from seller k to buyers is constant. If $p(R'_i) \geq(\leq)\ p(R_i)$, by the properties of the uniform rule (one-sided welfare domination under preference replacement in Thomson, 1997), for each buyer other that i, her assignment decreases (increases) or remains constant while being bounded above by her peak, thus making each buyer other than i at most (at least) as well-off going from $\varphi^4(R, A)$ to $\varphi^4(R', A)$ while the assignment of seller k remains unchanged at $d(R_k)$.

Case 2.2.2: $\min\{\sigma(R_B), d(R_k)\} = \sigma(R_B)$, $\min\{\sigma(R'_B), d(R_k)\} = \sigma(R'_B)$. Then, for each $j \in B \setminus \{i\}$, $\varphi^4_j(R, A) = p(R_j) = \varphi^4_j(R', A)$ and seller k may be made better or worse off.

Case 2.2.3: $\min\{\sigma(R_B), d(R_k)\} = d(R_k)$, $\min\{\sigma(R'_B), d(R_k)\} = \sigma(R'_B)$. Then, $p(R_k) \leq \sigma(R'_B) \leq \sigma(R_B)$ so k is at least as well-off and, for each $j \in B \setminus \{i\}$, $\varphi^4_j(R', A) = p(R_j)$ so each buyer other than i is made at least as well-off.

Case 2.2.4: $\min\{\sigma(R_B), d(R_k)\} = \sigma(R_B)$, $\min\{\sigma(R'_B), d(R_k)\} = d(R_k)$. Then,

$p(R_k) \leq \sigma(R_B) \leq \sigma(R'_B)$ so k is at most as as well-off and, for each $j \in B \setminus \{i\}$, $\varphi^4_j(R, A) = p(R_j)$ so each buyer other than i is made at most as well-off.

Thus, in all of the above cases, each agent other than i is affected in the same direction welfare-wise, as required by *replacement-dominance*.

We now prove that φ^3 does not satisfy *replacement-dominance*. Consider the economy e_1 introduced in part (ii) of this proof and again let $x \equiv \varphi^3(e_1)$. Assume additionally that $d(R_k) = d(R_l) = 2$. Then, since the sum of the peaks of the buyers (6) is greater than the sum of the peaks of the sellers (2) and $f(R) = |S| \cdot d(R_k) = 4$, the amount transferred from sellers to buyers is 4. Then, $x_i = x_j = x_k = x_l = 2$. Consider an economy e_2 identical to e_1 with the exception that seller k now has a preference relation R'_k with $p(R'_k) = 1$ as before but with $d(R'_k) = 1.5$ and let $y \equiv \varphi^3(e_2)$. Then, since the sum of the peaks of the buyers (6) is greater than the sum of the peaks of the sellers (2) and $f(R'_k, R_{\{i,j,l\}}) = |S| \cdot d(R'_k) = 3$, the amount transferred from sellers to buyers is 3. Then, $y_i = y_j = y_k = y_l = 1.5$. Thus, i and j are worse off while l is better off, in violation of *replacement-dominance*.

(iv) It is clear that rules $\varphi^1, \varphi^2, \varphi^3$, and φ^5 satisfy *replication invariance*. To see that φ^4 does not, let $B = \{i\}$, $S = \{j\}$, and $(R, A) \in \mathcal{E}^{B \cup S}$ be such that $p(R_i) = 4$, $p(R_j) = 1$, $d(R_j) = 3$, and $x \equiv \varphi^2(R, A)$. Then, since the sum of the peaks of the buyers (4) is greater than the sum of the peaks of the sellers (1) and $f(R) = d(R_j) = 3$, the amount transferred from sellers to buyers is 3. Then, $x_i = 3$ and $x_j = 3$. Since the duplicate of (R, A) involves two sellers and φ^4 coincides with the egalitarian rule in this case, $\varphi^2(2 * R, 2 * A) = E(2 * R, 2 * A) \equiv y$ where $y_i = 1 = y_{i^*}$ and $y_j = 1 = y_{j^*}$. Thus, $2 * x \neq y$.

(v) Note that, under rules φ^1 and φ^2, each agents' assignment is bounded above by her peak. In this range, agents' welfare is increasing above that of receiving 0. Thus, these rules satisfy *voluntary participation*.

We now prove that φ^3 satisfies *voluntary participation*. Let $N \equiv B \cup S \in \mathcal{N}$, $(R, A) \in \mathcal{E}^N$, and $x \equiv \varphi^3(R, A)$. Since φ^3 treats buyers and sellers symmetrically, assume without loss of generality that $\sigma(R_S) \leq \sigma(R_B)$. If $\sigma(R_S) = \sigma(R_B)$, each agent is assigned her peak which is welfare-wise superior to receiving 0. Suppose instead that $\sigma(R_S) < \sigma(R_B)$. Then, the amount transferred from sellers to buyers is

$$m = \text{median}\{\sigma(R_S), \sigma(R_B), |S|d\} \quad \text{where } d \equiv \min_{i \in S} d(R_i), \tag{10}$$

for each $i \in B$, $x_i \leq p(R_i)$, and, for each $i \in S$, $x_i \geq p(R_i)$. Thus, by single-peakedness, each buyer's assignment under x is at least as desirable as 0. It remains to show that the same is true for sellers. By definition, for each $i \in S$, $x_i = U_i(R_S, m) = \max\{\lambda, p(R_i)\}$ where $\lambda \in \mathbb{R}_+$ is such that $\sum_S \max\{\lambda, p(R_i)\} = m$. Thus, letting

$S' \equiv \{i \in S : p(R_i) > \lambda\}$, for each $i \in S'$, $x_i = p(R_i)$, and, for each $i \in S \setminus S'$, $x_i = \lambda$. If $m = \sigma(R_S)$, then each seller receives her peak assignment and we are done so suppose instead that $m \neq \sigma(R_S)$. By (10), this in turn requires $m = \min\{\sigma(R_B), |S|d\} > \sigma(R_S)$.

- Case 1: S' is empty. Then, for each $i \in S$, $x_i = \lambda = \frac{m}{|S|} \leq \frac{|S|d}{|S|} = d \leq d(R_i)$.
- Case 2: S' is non-empty. Then, $S \setminus S'$ is also non-empty. Since m is the total amount transferred by sellers to buyers, defining $p \equiv \frac{\sum_{S'} p(R_i)}{|S'|}$ to be the average assignment received by sellers in S' and recalling that λ is the assignment received by sellers in $S \setminus S'$, we have that $m = |S'|p + |S \setminus S'|\lambda$ or, rearranging, $\lambda = \frac{m - |S'|p}{|S \setminus S'|}$. Since, $p(R_i) > \lambda$ for each $i \in S'$, we have $p > \lambda$. Now, $m = |S'|p + |S \setminus S'|\lambda$ and $p > \lambda$ imply that $p > \frac{m}{|S|}$. Thus, since $m = \min\{\sigma(R_B), |S|d\}$, $\lambda = \frac{m - |S'|p}{|S \setminus S'|} < \frac{m - |S'|\frac{m}{|S|}}{|S \setminus S'|} = \frac{m}{|S|} \leq \frac{|S|d}{|S|} = d$. Thus, for each $i \in S \setminus S'$, $x_i = \lambda < d \leq d(R_i)$, and, for each $i \in S'$, $x_i = p(R_i)$.

In both cases above, by single-peakedness, each agent's assignment under x is at least as desirable as 0. Thus, φ^3 satisfies *voluntary participation*.

We now prove that φ^4 satisfies *voluntary participation*. In economies with more than one seller, φ^4 coincides with the egalitarian rule which satisfies *voluntary participation*. Thus, it suffices to establish that each agent finds her assignment to be at least as desirable as 0 when the set of sellers contains exactly one seller. Let $N \equiv B \cup S \in \mathcal{N}$ be such that S consists of a single seller denoted by i, $(R, A) \in \mathcal{E}^N$, and $x \equiv \varphi^4(R, A)$. If $\sigma(R_S) < \sigma(R_B)$, the amount transferred from the seller to the buyers is $m \equiv \text{median}\{\sigma(R_S), \sigma(R_B), d(R_i)\} = \min\{\sigma(R_B), d(R_i)\}$ and, by definition, $x_i = m \leq d(R_i)$ and, for each $j \in B$, $x_j \leq p(R_j)$. If $\sigma(R_S) \geq \sigma(R_B)$, the amount transferred from the seller to the buyers is $m \equiv \text{median}\{\sigma(R_S), \sigma(R_B), 0\} = \sigma(R_B)$ and, by definition, $x_i = m = \sigma(R_B) \leq \sigma(R_S) = d(R_i)$ and, for each $j \in B$, $x_j = p(R_j)$. In both cases, by single-peakedness, each agent's assignment under x is at least as desirable as 0. Thus, φ^4 satisfies *voluntary participation*.

To see that φ^5 does not satisfy *voluntary participation*, consider an economy where each buyer has a peak of 0 and there is at least one seller with a strictly positive peak. Under φ^5, this seller is awarded her peak. Since the sum of sellers' assignments is equal to the sum of buyers', there is a buyer who is assigned a positive amount. This makes her worse off than receiving 0.

(vi) The only preference information rules φ^1, φ^2, and φ^5 rely on is the profile of peaks. Thus, for any two preference profiles with the same peaks, these rules recommend the same allocation. These rules are thus *peaks-only*. Rules φ^3 and φ^4 are not since they are sensitive to further preference information. ∎

References

Bochet, O., R. İlkılıç, and H. Moulin (2013). Egalitarianism under earmark constraints. *Journal of Economic Theory 148*, 535–562.

Bochet, O., R. İlkılıç, H. Moulin, and J. Sethuraman (2012). Balancing supply and demand under bilateral constraints. *Theoretical Economics 7*, 395–423.

Bogomolnaia, A. and H. Moulin (2004). Random matching under dichotomous preferences. *Econometrica 72*(1), 257–279.

Ching, S. (1994). An alternative characterization of the uniform rule. *Social Choice and Welfare 11*(2), 131–136.

Chun, Y. (2006). The separability principle in economies with single-peaked preferences. *Social Choice and Welfare 26*(2), 239–253.

Dagan, N. (1996). A note on Thomson's characterization of the uniform rule. *Journal of Economic Theory 69*(1), 255–261.

De Frutos, A. and J. Massó (1995). More on the uniform allocation rule: equality and consistency. WP 288.95 UAB, 1995.

Dutta, B. and D. Ray (1989). A concept of egalitarianism under participation constraints. *Econometrica 57*(3), 615–635.

Foley, D. (1967). Resource allocation and the public sector. *Yale Economic Essays 7*.

Fujishige, S. (2005). *Submodular Functions and Optimization*, Volume 58 of *Annals of Discrete Mathematics*. Elsevier.

Gibbard, A. (1985). Manipulation of voting schemes: a general result. *Econometrica 41*(4), 587–601.

Goswami, M., M. Mitra, and A. Sen (2013). Strategy-proofness and pareto-efficiency in quasi-linear exchange economies. *Theoretical Economics*.

Hardy, G., J. Littlewood, and G. Polya (1929). Some simple inequalities satisfied by convex functions. *Messenger of Mathematics 58*.

Hurwicz, L. (1972). On informationally decentralized systems. In C. McGuire and R. Radner (Eds.), *Decision and Organization*, pp. 297–336. North-Holland.

Kesten, O. (2006). More on the uniform rule: characterizations without Pareto-optimality. *Mathematical Social Sciences 51*, 192–200.

Kıbrıs, O. and S. Küçükşenel (2009). Uniform trade rules for uncleared markets. *Social Choice and Welfare 32*, 101–121.

Klaus, B., H. Peters, and T. Strocken (1998). Strategy-proof division with single-peaked preferences and individual endowments. *Social Choice and Welfare 15*.

Lovász, L. and M. Plummer (1986). *Matching Theory*. North Holland.
Moulin, H. (1987). The pure compensation problem: Egalitarianism versus laissez-fairism. *Quarterly Journal of Economics 102*, 769–783.
Moulin, H. (1999). Rationing a commodity along fixed paths. *Journal of Economic Theory 84*(1), 41–72.
Roth, A., T. Sönmez, and U. Ünver (2005). Pairwise kidney exchange. *Journal of Economic Theory 125*, 151 – 188.
Satterthwaite, M. (1975). Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare theorems. *Journal of Economic Theory 10*, 187–217.
Schmeidler, D. (1979). A bibliographical note on a theorem by Hardy, Littlewood, and Polya. *Journal of Economic Theory 59*(20), 125–128.
Schrijver, A. (2003). *Combinatorial Optimiazation*. Springer.
Serizawa, S. (2002). Inefficiency of strategy-proof rules for pure exchange economies. *Journal of Economic Theory 106*(2), 219–241.
Sönmez, T. (1994). Consistency, monotonicity, and the uniform rule. *Economics Letters 46*, 229 – 235.
Sprumont, Y. (1991). The division problem with single-peaked preferences. *Econometrica 59*(2), 509–519.
Thomson, W. (1994a). Consistent solutions to the problem of fair division when preferences are single-peaked. *Journal of Economic Theory 63*(2), 219 – 245.
Thomson, W. (1994b). Resource-monotonic solutions to the problem of fair division when preferences are single-peaked. *Social Choice and Welfare 11*, 205 – 223.
Thomson, W. (1995a). Axiomatic analysis of generalized economies with single-peaked preferences. revised (2009).
Thomson, W. (1995b). Population-monotonic solutions to the problem of fair division when preferences are single-peaked. *Economic Theory 5*(2), 229 – 246.
Thomson, W. (1997). The replacement principle in economies with single-peaked preferences. *Journal of Economic Theory 76*(1), 145 – 168.
Thomson, W. (1999). Welfare-domination and preference-replacement: a survey and open questions. *Social Choice and Welfare 16*, 373 – 394.
Thomson, W. (2011). Fair allocation rules. In K. Arrow, A. Sen, and K. Suzumura (Eds.), *Handbook of Social Choice and Welfare*, Volume 2, pp. 393–506. Elsevier.
Zhou, L. (1991). Inefficiency of strategy-proof allocation mechanisms in pure exchange economies. *Social Choice and Welfare 8*, 247 – 254.